Math 210A Discussion Week 1

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Fall 2014 Problem 6. Let G be a finite group and let p be the smallest prime number dividing the order of G. Assume G has a normal subgroup H of order p. Show that H is contained in the center of G.

Conjugating elements of H by G is a group action since H is a normal subgroup. The fixed points of the action are exactly the elements of H in $Z(G)$. Thus $p = |H| = |Z(G) \cap H| + \sum_{h \notin Z(G)} |\mathrm{Orb}(h)|$. The identity is contained in H and $Z(G)$ which implies $|H \cap Z(G)| \geqslant 1$ and $|\mathrm{Orb}(h)| < p$ for all $h \notin Z(G)$. Orbit-Stabilizer gives us $|\mathrm{Orb}(h)| = [G : \mathrm{Stab}(h)]$ so $|\mathrm{Orb}(h)|$ divides $|G|$. Since p is the smallest prime that divides $|G|$, we conclude there are no elements $h \notin Z(G)$. Thus $H \subset Z(G)$.

Spring 2016 Problem 9. Show that if G is a finite group acting transitively on a set X with at least two elements, then there exists $g \in G$ which fixes no point of X.

Let $n = |G|$ and $k = |X| \geqslant 2$. Define $\mathrm{Fix}(g) = \{x \in X : g \cdot x = x\}$. For each $g \in \mathrm{Stab}(x)$, we have $x \in \mathrm{Fix}(g) = \{x \in X : gx = x\}$ and visa versa. We conclude $\sum_{x \in X} |\mathrm{Stab}(x)| = \sum_{g \in G} |\mathrm{Fix}(g)|$. By Orbit-Stabilizer and $|G|$ finite, $|\mathrm{Stab}(x)| = |G|/|\mathrm{Orb}(x)|$ for all $x \in X$. But G acts transitively on X so $|\mathrm{Orb}(x)| = |X| = k$ and $|\mathrm{Stab}(x)| = \frac{n}{k}$. Then $\sum_{g \in G} |\mathrm{Fix}(g)| = \sum_{x \in X} \frac{n}{k} = n$. Since $|\mathrm{Fix}(e)| = k \geqslant 2$, we have $\sum_{g \in G, g \neq e} |\mathrm{Fix}(g)| < n - 1$. If all non-identity $g \in G$ have $|\mathrm{Fix}(g)| = 1$, we would have $\sum_{g \in G, g \neq e} |\mathrm{Fix}(g)| = n - 1$. By the pigeonhole principle, there is some g such that $|\mathrm{Fix}(g)| = 0$ as desired.

Fall 2018 Problem 1. Let $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ be the quaternion group of order 8.

(a) Show that every non-trivial subgroup of Q_8 contains -1.

Let $H \subset Q_8$ be a non-trivial subgroup. If $-1 \in H$, then we are done. Otherwise, one of $\{\pm i, \pm j, \pm k\}$ is in H. But $(\pm i)^2 = (\pm j)^2 = (\pm k)^2 = -1 \in H$. Therefore, each non-trivial subgroup of Q_8 contains -1.

(b) Show that Q_8 does not embed in the symmetric group S_7 (as a subgroup).

Let $\phi : Q_8 \to S_7$ be an injective group homomorphism. This defines a group action of Q_8 on the set $X = \{x_1, \ldots, x_7\}$ via $g \cdot x_i = x_{\phi(g)(i)}$ for $g \in Q_8$. The orbits of the action partition X so $|X| = \sum_{x \in X} |\mathrm{Orb}(x)|$. By Orbit-Stabilizer, $|\mathrm{Orb}(x)| = [Q_8 : \mathrm{Stab}(x)] = |Q_8|/|\mathrm{Stab}(x)|$ by $|Q_8|$ finite. Note $\mathrm{Stab}(x)$ is a non-trivial subgroup of Q_8 for all $x \in X$ since $|Q_8|/|\mathrm{Stab}(x)| = 8 > 7$, a contradiction. By (a), $-1 \in \mathrm{Stab}(x)$ for all $x \in X$ so $\phi(-1) = e$. This contradicts the injectivity of ϕ.

Spring 2019 Problem 8. Prove that every finite group of order n is isomorphic to a subgroup of $\mathrm{GL}_{n-1}(\mathbb{C})$.

By Cayley's Theorem, there is an injective homomorphism from G to S_n. There is an injective homomorphism S_n to $\mathrm{GL}_n(\mathbb{C})$ given by permuting the elements of $\mathbb{C}^n$ once a basis has been chosen. Let $v \in \mathbb{C}^n$ be the vector of all 1s, which is an eigenvector for each permutation matrix. Each permutation matrix in the basis $\beta = \{v, e_2, \ldots, e_n\}$ for $\mathbb{C}^n$ will be a block matrix of (1) and a permutation matrix in $\mathrm{GL}_{n-1}(\mathbb{C})$. Thus there is an injective homomorphism of S_n to $\mathrm{GL}_{n-1}(\mathbb{C})$. Compose this with the injection from Cayley's Theorem to prove the claim.

Spring 2020 Problem 7. Let G be a **finite** p-group and $1 \neq N \subset G$ be a non-trivial normal subgroup.

(a) Show that N contains a non-trivial element of the center $Z(G)$ of G.

Conjugating elements of N by G is a group action since N is a normal subgroup. The fixed points of the action are exactly the elements of N in $Z(G)$. Thus $|N| = |Z(G) \cap N| + \sum_{h \in N, h \notin Z(G)} |\mathrm{Orb}(h)|$. The identity is contained in N and $Z(G)$ which implies $|N \cap Z(G)| \geqslant 1$. By Orbit-Stabilizer, $|\mathrm{Orb}(h)| = |G|/|\mathrm{Stab}(h)|$, which is divisible by p. Then $|N| - \sum_{h \in N, h \notin Z(G)} |\mathrm{Orb}(h)| = |Z(G) \cap N|$ is divisible by p, and there is some non-trivial element of $Z(G) \cap N$.

We will cover the non-finite case when we talk about Sylow p-subgroups.

(b) Give an example where $Z(N) \not\subset Z(G)$.

Take $G = D_4$, the dihedral group of order 8. Let $N = \langle r \rangle$ be the cyclic subgroup of G generated by rotation by $\frac{\pi}{2}$ counter-clockwise. Then $Z(N) = N$ but $Z(G) = \langle r^2 \rangle$.

Math 210A Discussion Week 3

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A *free group* is a group containing all words on a set of letters S. Given a function f from S to G, there exists a unique group homomorphism $\varphi : F \to G$ such that φ restricts to f on X.

Spring 2017 Problem 3. Find the number of subgroups of index 3 in the free group $F_2 = \langle u, v\rangle$ on two generators. Justify your answer.

Let $X = \{1, 2, 3\}$ be a set of order 3. Assume there is a transitive group action of F_2 on X. Then Stab(1) is a subgroup of G with $[G : \text{Stab}(1)] = |\text{Orb}(1)| = 3$ by Orbit-Stabilizer. Now assume H is an index 3 subgroup of F_2. Then the set F_2/H of left cosets has order 3. We have a transitive group action of F_2 on the set F_2/H given by left multiplication. Let $g \in F_2$. We have $g \cdot H = H$ if and only if $g \in H$. As a result, $\text{Stab}(H) = H$. The two situations describe a bijection between index 3 subgroups of G and stabilizers of transitive group actions on sets of three elements.

We will find the number of transitive group actions of F_2 on the set $X = \{1, 2, 3\}$ with $H := \text{Stab}(1)$. In the case of $|X| = 3$, this is equivalent to finding a homomorphism $\phi : F_2 \to S_3$ whose image contains a 3-cycle. The image of u and v under ϕ uniquely determines ϕ by the universal property of free groups. We will break into cases. Note that 2 and 3 can are interchangeable so $\phi(u) = (13)$ cases produce the same stabilizers of 1 as the $\phi(u) = (12)$ cases. Similarly, we do not have to consider $\phi(u) = (132)$.

$$\begin{aligned}
&\phi(u) = e \text{ implies } \phi(v) \in \{(123), (132)\}\\
&\phi(u) = (12) \text{ implies } \phi(v) \in \{(13), (23), (123), (132)\}\\
&\phi(u) = (23) \text{ implies } \phi(v) \in \{(12), (13), (123), (132)\}\\
&\phi(u) = (123) \text{ implies } \phi(v) \in \{e, (12), (13), (23), (123), (132)\}
\end{aligned}$$

The symmetry of 2 and 3 also allows us to remove the cases $\{\phi(u) = e, \phi(v) = (132)\}$, $\{\phi(u) = (23), \phi(v) = (13)\}$, and $\{\phi(u) = (23), \phi(v) = (132)\}$. We are left with 13 suitable group homomorphisms $\phi : F_2 \to S_3$ for which Stab(1) determines all distinct subgroups of F_2 of index 3.

Let S be a subset of a group G. The *normalizer* $N_G(S) = \{g \in G : gSg^{-1} = S\}$. We can prove that $N_G(S)$ is a subgroup of G. Note that for $S = H$, a subgroup of G, H is a normal subgroup of $N_G(H)$.

The *commutator subgroup* $[G, G]$ of a group G is the subgroup generated by $ghg^{-1}h^{-1}$ for $g, h \in G$. It is the smallest subgroup of G such that $G/[G, G]$ is an abelian group. In other words, G/N is abelian if and only if N contains $[G, G]$.

Fall 2017 Problem 2. Let G be a finite group of order a power of a prime number p. Let $\Phi(G)$ be the subgroup of G generated by elements of the form g^p for $g \in G$ and $ghg^{-1}h^{-1}$ for $g, h \in G$. Show that $\Phi(G)$ is the intersection of the maximal proper subgroups of G.

Let G be a p-group that acts on a finite set X. We will first show that $|X^G| \equiv |X| \pmod p$ where

$$X^G = \{x \in X : |\text{Orb}(x)| = 1\}.$$

The orbits partition X so

$$|X| = |X^G| + \sum_{x \in X, x \notin X^G} |\text{Orb}(x)|.$$

By Orbit-Stabilizer, $|\mathrm{Orb}(x)| = [G : \mathrm{Stab}(x)] = |G|/|\mathrm{Stab}(x)|$ with $|G|$ finite. For each $x \notin X^G$, we have

$$|\mathrm{Orb}(x)| = |G|/|\mathrm{Stab}(x)| > 1$$

so p will divide $|G|/|\mathrm{Stab}(x)| = |\mathrm{Orb}(x)|$. Therefore, $|X| \equiv |X^G|$ modulo p.

Let $|G| = p^k$. Let $H \subset G$ be a maximal proper subgroup of G so $|H| = p^{k-1}$. Let H act on the left cosets of H in G by left multiplication. If $aH \in X^H$, then $b(aH) = aH$ for all $b \in H$. Thus $aba^{-1} \in H$ and $a \in N_G(H)$. Similarly, taking some $a \in N_G(H)$ gives $aH \in X^H$. Therefore, $X^H = [N_G(H) : H]$ and the above result implies $[N_G(H) : H] \equiv [G : H] \equiv 0 \pmod p$. Then index $[N_G(H) : H]$ divides $[G : H]$ so it is either 1 or p. We conclude that $[N_G(H) : H] = p$ and $N_G(H) = G$ since $|H| = p^{k-1}$. Thus H is a normal subgroup of G so the set G/H is a group of order p. The only such group is the cyclic group $\mathbb{Z}/p\mathbb{Z}$ so $G/H \simeq \mathbb{Z}/p\mathbb{Z}$. If $g \notin H$ for $g \in G$, then gH is a generator of G/H so $(gH)^p = g^pH = H$. H contains elements of the form g^p for $g \in G$. Further, G/H is abelian so the canonical projection $p : G \to G/H$ factors through $\pi : G/[G,G]$ for $[G,G]$ the commutator subgroup. Thus $\ker(\pi) = [G,G] \subset \ker(p) = H$ and H contains all elements of the form $ghg^{-1}h^{-1}$ for $g, h \in H$. Therefore, $\Phi(G)$ is contained in the intersection of the maximal proper subgroups of G.

For each $g \notin \Phi(G)$, we want to show that there is a maximal proper subgroup $M \subset G$ that does not contain g. The commutator subgroup of G is normal. Let $g, h \in G$. Then $hg^ph^{-1} = (hgh^{-1})^p \in \Phi(G)$ so $\Phi(G)$ is a normal subgroup of G. Every element $g \in G$ with $g \notin \Phi(G)$ corresponds to a coset $\overline{g} = g\Phi(G) \in G/\Phi(G)$. By $g^p \in \Phi(G)$ for all $g \in G$, $G/\Phi(G)$ is a group where each element divides order p. Since the commutator subgroup is contained in $\Phi(G)$, $G/\Phi(G)$ is a finite abelian group with only elements of order dividing p. We can view $G/\Phi(G)$ as an F_p-vector space so take an F_p-basis $\{\overline{g}, \overline{x_1}, \ldots, \overline{x_k}\}$ for $G/\Phi(G)$. Let x_i be a lift of $\overline{x_i}$ in G. Define the subgroup M generated by $\Phi(G) \cup \{x_1, \ldots, x_k\}$. Since $g \notin M$ by construction, M is a proper subgroup of G. Further, $M \cup \{g\} = G$ so G is a maximal proper subgroup of G that does not contain g. We conclude that the intersection of the maximal proper subgroups of G is contained in $\Phi(G)$.

The set of automorphisms of a group G forms a group, denoted $\mathrm{Aut}(G)$. The set of inner automorphisms, those represented by conjugation by some $g \in G$, is a subgroup of $\mathrm{Aut}(G)$. We denote these $\mathrm{Inn}(G)$.

Fall 2018 Problem 2. Let G be a finitely generated **infinite** group having a subgroup of finite index $n > 1$. Show that G has finitely many subgroups of index n and has a proper characteristic subgroup (i.e. preserved by all automorphisms) of finite index.

There are finite groups for which the statement does not hold. Conjugation by an element of a group is an automorphism of the group (called an inner automorphism). Thus every characteristic subgroup of a group is normal. The finite group A_5 is simple and thus contains no non-trivial characteristic subgroups. Assume G is infinite.

Let $H \subset G$ be a subgroup of index n. Then G acts on the set of left cosets $G/H = \{g_1H, g_2H, \ldots, g_nH\}$ via left multiplication. This defines a group homomorphism $\phi : G \to S_n$ such that $g \cdot g_iH = g_{\phi(g)(i)}H$. Note that $g \cdot H = H$ if and only if $g \in H$. Thus $\mathrm{Stab}(H) = H$ implying a one-to-one correspondence between the index n subgroups of G and homomorphisms $\phi : G \to S_n$. Let G be finitely generated by $\{x_1, \ldots, x_k\}$, say. Then the image of each x_i in S_n determine uniquely each homomorphism $\phi : G \to S_n$. There are $n!$ choices for the image of each x_i so there are finitely many homomorphisms $\phi : G \to S_n$. We conclude there are finitely many index n subgroups of G.

Let $\sigma \in \mathrm{Aut}(G)$ and $H \subset G$ be the index n subgroup in the problem statement. Now $\sigma(H)$ is a subgroup of G since σ is an automorphism. Note that the cosets are $\sigma(G)/\sigma(H) = G/\sigma(H) = \{\sigma(g_1)\sigma(H), \ldots, \sigma(g_n)\sigma(H)\}$ so $\sigma(H)$ is an index n subgroup of G. Define $N := \bigcap_{\sigma \in \mathrm{Aut}(G)} \sigma(H)$. There are finitely many index n subgroups of G so $N = \bigcap_{i=1}^m H_i$ for some index n subgroups $H_i \subset G$. We want to show that N is a proper characteristic subgroup of finite index in G. It is clear that N is a subgroup that is fixed under all automorphisms of G. We can define a group action of G on $\prod_{i=1}^m G/H_i$ by component-wise left multiplication. Then $\mathrm{Stab}(H_1, H_2, \ldots, H_m) = \bigcap_{i=1}^m H_i = N$ since $gH_i = H_i$ if and only if $g \in H_i$. By Orbit-Stabilizer,

$$[G : N] = [G : \mathrm{Stab}(H_1, H_2, \ldots, H_m)] = |\mathrm{Orb}(H_1, H_2, \ldots, H_m)| \leqslant |\mathrm{Orb}(H_1)| \cdots |\mathrm{Orb}(H_n)| = [G : H_1] \cdots [G : H_m].$$

Since each H_i is of finite index, $[G : N]$ is finite. Therefore, N is a characteristic subgroup of G of finite index. Note that N cannot be all of G since it is a subgroup of a H and N is not trivial since it is a finite index subgroup of an infinite group.

Fall 2015 Problem 8. Let F be a field. Show that the group $\mathrm{SL}(2, F)$ is generated by the matrices $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$ for elements e in F.

The group $\mathrm{SL}(2, F)$ is all 2×2 contains matrices with determinant one. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ be a general matrix in $\mathrm{SL}(2, F)$. Case 1: If $a = 0$ or $d = 0$, then $c = -b^{-1}$.

$$\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -e^{-1} & 1 \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e^{-1} & 1 \end{pmatrix}$$
$$\begin{pmatrix} 0 & e \\ -e^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & e(1-a) \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & e \\ -e^{-1} & a \end{pmatrix}$$
$$\begin{pmatrix} 1 & 0 \\ -e^{-1} & 1 \end{pmatrix}\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e \\ -e^{-1} & 0 \end{pmatrix}$$
$$\begin{pmatrix} 1 & e(1-a) \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & e \\ -e^{-1} & 0 \end{pmatrix} = \begin{pmatrix} a & e \\ -e^{-1} & 0 \end{pmatrix}$$

Case 2: If $b = 0$ or $c = 0$, then $d = a^{-1}$.

$$\begin{pmatrix} 0 & -be \\ e^{-1}b^{-1} & 0 \end{pmatrix}\begin{pmatrix} 0 & e \\ -e^{-1} & -e^{-1}b^{-1}a \end{pmatrix} = \begin{pmatrix} b & a \\ 0 & b^{-1} \end{pmatrix}$$
$$\begin{pmatrix} 0 & -e^{-1} \\ e & -e^{-1}b^{-1}a \end{pmatrix}\begin{pmatrix} 0 & e^{-1}b^{-1} \\ -be & 0 \end{pmatrix} = \begin{pmatrix} b & 0 \\ a & b^{-1} \end{pmatrix}$$

Case 3: Assuming nonzero $a, b, c, d \in F$, then $A = \begin{pmatrix} d^{-1}(1+bc) & b \\ c & d \end{pmatrix}$.

$$\begin{pmatrix} b & a \\ 0 & b^{-1} \end{pmatrix}\begin{pmatrix} c & 0 \\ d & c^{-1} \end{pmatrix} = \begin{pmatrix} bc + ad & ac^{-1} \\ b^{-1}d & b^{-1}c^{-1} \end{pmatrix}$$

Then $(b^{-1}c^{-1})^{-1}(1 + (ac^{-1})(b^{-1}d)) = bc(1 + ab^{-1}c^{-1}d) = bc + ad$, the first row, first column entry above. We conclude $\mathrm{SL}(2, F)$ is generated by $\begin{pmatrix} 1 & e \\ 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 0 \\ e & 1 \end{pmatrix}$ for $e \in F$.

Fall 2015 Problem 10. Let p be a prime number. For each abelian group K of order p^2, how many subgroups H of $\mathbb{Z}^3$ are there with $\mathbb{Z}^3/H$ isomorphic to K.

Note that $\mathbb{Z}^3$ is abelian so each subgroup $H \subset \mathbb{Z}^3$ is normal. Let S be the set of surjective group homomorphisms $f : \mathbb{Z}^3 \to K$ and T be the set of all subgroups H of $\mathbb{Z}^3$ for which $\mathbb{Z}^3/H \simeq K$. Then define a set map $\Phi : S \to T$ by $\Phi(f) = \ker(f)$. Let $\mathrm{Aut}(K)$ be the group automorphism of K, and $\mathrm{Aut}(K)$ acts on S by post-composition. Denote by $S/\mathrm{Aut}(K)$ the set of orbits of S under the action by $\mathrm{Aut}(K)$. Let $\sigma \in \mathrm{Aut}(K)$, then $\ker(\sigma \circ f) = \ker(f)$ since σ is injective. As a result, $\overline{\Phi} : S/\mathrm{Aut}(K) \to T$ is a well-defined set map. Surjectivity of $\overline{\Phi}$ follows from the fact that each subgroup H for which $\mathbb{Z}^3/H \simeq K$ defines a surjective group homomorphism $f : \mathbb{Z}^3 \to \mathbb{Z}^3/H \simeq K$.

We want to show that $\overline{\Phi}$ is injective. Let $f, g \in S$ such that $\ker(f) = \ker(g)$. By the universal property of quotients, f factors through $\mathbb{Z}^3/\ker(f)$, and there is some isomorphism $\alpha : \mathbb{Z}^3/\ker(f) \to K$ such that $\alpha \circ \pi = f$ for $\pi : \mathbb{Z}^3 \to \mathbb{Z}^3/H$ the canonical quotient homomorphism. Similarly, $\beta \circ \pi = g$ for an isomorphism $\beta : \mathbb{Z}^3/\ker(g) \to K$. Then $f = (\alpha \circ \beta^{-1}) \circ g$ where $(\alpha \circ \beta^{-1}) \in \mathrm{Aut}(K)$, and f and g are in the same $\mathrm{Aut}(K)$-orbit of S. We conclude that $\overline{\Phi}$ is a bijection.

It is sufficient to find the number of surjective group homomorphisms $f : \mathbb{Z}^3 \to K$ for each K. There are only two abelian groups of order p^2: $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Case 1: Let $K = \mathbb{Z}/p^2\mathbb{Z}$. We need only find images for the 3 generators of the free abelian group $\mathbb{Z}^3$. Let $x, y \in \mathbb{Z}/p^2\mathbb{Z}$ be non-generating elements. They are classes represented by integers divisible by p. Then representatives of $x + y$ are divisible by p and $x + y$ does not generate $\mathbb{Z}/p^2\mathbb{Z}$. Thus at least one of the generators of $\mathbb{Z}^3$ must map to a generator of $\mathbb{Z}/p^2\mathbb{Z}$ in order for the homomorphism to be surjective. There are $\phi(p^2) = p^2 - p$ generators of $\mathbb{Z}/p^2\mathbb{Z}$ for Euler's totient function φ. There are p^6 total homomorphisms and p^3 homomorphisms that are not surjective. Since $|\mathrm{Aut}(\mathbb{Z}/p^2\mathbb{Z})| = \varphi(\mathbb{Z}/p^2\mathbb{Z}) = p^2 - p$, there are $\frac{p^6-p^3}{p^2-p} = p^4 + p^3 + p^2$ total subgroups H of $\mathbb{Z}^3$ for which $\mathbb{Z}^3/H \simeq \mathbb{Z}/p^2\mathbb{Z}$.

Case 2: Let $K = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$. Once again, we need only find images for the 3 generators of the free abelian group $\mathbb{Z}^3$. Note that K is no longer generated by just one element. For the homomorphism to be surjective, we need the image of at least two of the generators of $\mathbb{Z}^3$ to map to generators of K. This equates to sending one generator to a nontrivial element $a \in K$ and sending a second to an element outside the subgroup generated by a in K. The subgroup generated by a will have order p. We have three scenarios. If the first generator is sent to a nonzero $a \in K$, we have $(p^2-1)(p^2-p)(p^2) + (p^2-1)(p)(p^2-p)$ options depending on the image of the second generator. If the first generator is sent to zero, we have $(p^2-1)(p^2-p)$ options. In total, we have $p^6 - p^4 - p^3 + p$ surjective homomorphisms. There are $(p^2-1)(p^2-p) = p^4 - p^3 - p^2 + p$ automorphisms of K which implies $\frac{p^6-p^4-p^3+p}{p^4-p^3-p^2+p} = p^2 + p + 1$ subgroups H of $\mathbb{Z}^3$ such that $\mathbb{Z}^3/H \simeq \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Spring 2017 Problem 1, Fall 2019 Problem 6. Choose a representative for every conjugacy class in the group $\mathrm{GL}(2, \mathbb{R})$. Justify your answer.

Each conjugacy class of matrices in $\mathrm{GL}(2, \mathbb{R})$ has a unique representative in rational canonical form. For 2×2 matrices, the invariant factors of $A \in \mathrm{GL}(2, \mathbb{R})$ could be $\{f\}$ for $f = x^2 - ax - b \in \mathbb{R}[x]$ or $\{g, h\}$ where $g|h$. Since the sum of the degrees of g and h is 2, we see that $\deg(g) = \deg(h) = 1$. We can take g and h monic so $g = h = x - c$ for some $c \in \mathbb{R}$. Thus the possible rational canonical forms for a matrix in $\mathrm{GL}(2, \mathbb{R})$ are

$$\begin{pmatrix} 0 & b \\ 1 & a \end{pmatrix} \text{ or } \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}$$

for $a, b, c \in \mathbb{R}$. Each conjugacy classes of $\mathrm{GL}(2, \mathbb{R})$ has a representative of the form above.

Math 210A Discussion Week 4

Matthew Gherman

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We will briefly review Sylow's Theorems and semi-direct products. Let G be a finite group with $|G| = p^n m$ where $n \geqslant 1$ and $\gcd(p, m) = 1$.

(1) For every prime factor p of $|G|$, there exists a Sylow p-subgroup of G.

(2) All Sylow p-subgroups are conjugate by some element of G.

(3) Let n_p be the number of distinct Sylow p-subgroups of G. Then n_p divides m and $n_p \equiv 1 \pmod{p}$.

A *semi-direct product* of two groups H and K, denoted $H \rtimes_\varphi K$, is the direct product as a set with multiplication defined as $(h_1, k_1) \cdot (h_2, k_2) = (h_1 \varphi(k_1)(h_2), k_1 k_2)$ where $\varphi : K \to \mathrm{Aut}(H)$ is a group homomorphism. In particular, the homomorphism φ determines conjugation of an element in H by an element in K:

$$(e, k) \cdot (h, e) \cdot (e, k)^{-1} = (\varphi(k)(h), e).$$

If the homomorphism $\varphi : K \to \mathrm{Aut}(H)$ is trivial, the semi-direct product is the standard direct product. Possibly the most useful result about semi-direct products is as follows. If G is a group with a subgroup K and a normal subgroup H such that $G = HK$, then $G \simeq H \rtimes_\varphi K$ for some homomorphism $\varphi : K \to \mathrm{Aut}(H)$. As a result, the semi-direct product will play a central role in classifying finite groups.

Spring 2015 Problem 8. Let G be a finite group of order pq, where p and q are distinct primes. Show that

(a) G has a normal subgroup distinct from 1 and G

Without loss of generality, assume $p > q$. Let m_p denote the number of Sylow p-subgroups of G. By Sylow's Third Theorem, $m_p \equiv 1 \pmod{p}$ and m_p divides q. Since q is prime, m_p is either 1 or q. But $q \not\equiv 1 \pmod{p}$ since $p > q$. Thus $m_p = 1$. Conjugation of a subgroup $H \subset G$ by $g \in G$ is again a subgroup of G of order $|H|$. Thus we will obtain a Sylow p-subgroup of G when we conjugate a Sylow p-subgroup by any element $g \in G$. Since we have a unique Sylow p-subgroup $P \subset G$, $gPg^{-1} = P$ and P is normal in G.

(b) if $p \not\equiv 1 \pmod{q}$ and $q \not\equiv 1 \pmod{p}$, then G is abelian.

Without loss of generality, assume $p > q$. By (a), the Sylow p-subgroup $P \subset G$ is a normal subgroup of G. Sylow's Theorems imply the existence of some Sylow q-subgroup $Q \subset G$. The subgroup $P \cap Q$ is a subgroup of both P and Q. Then $|P \cap Q| = 1$ since $|P|$ and $|Q|$ are relatively prime. All of this implies $G = P \rtimes Q$ for some group homomorphism $\varphi : Q \to \mathrm{Aut}(P)$. We have $\mathrm{Aut}(P) \simeq \mathbb{Z}/(p-1)\mathbb{Z}$. The generator $a \in Q$ has order q so it needs to map to an element of order dividing q, leaving 1 or q. By assumption, $p \not\equiv 1 \pmod{q}$ so $\varphi(a)$ is the identity automorphism. Thus $G \simeq P \times Q$ for P, Q cyclic (which implies abelian). We conclude that G is abelian.

Fall 2015 Problem 5.

(a) Let G be a group of order $p^e v$ with v and e positive integers, p prime, $p > v$, and v not a multiple of p. Show that G has a normal Sylow p-subgroup.

By Sylow's Third Theorem, the number of Sylow p-subgroups m_p satisfies $m_p \equiv 1 \pmod{p}$ and m_p divides v. Thus $m_p = kp + 1$ for $k \geqslant 0$. However, $p > v$ and $m_p | v$ implies $k = 0$. We conclude $m_p = 1$. Let P be the unique Sylow p-subgroup of G. As in Spring 2015 Problem 8, conjugation of P by an element $g \in G$ is another Sylow p-subgroup. Thus $gPg^{-1} = P$ and P is a normal Sylow p-subgroup of G.

(b) Show that a nontrivial finite p-group has a nontrivial center.

Let H be a nontrivial finite p-group. Thus $|H| = p^k$ for $k > 0$. Act on the set H by H via conjugation. An element is fixed by conjugation if and only if the element is in the center of H. The class equation implies

$$|H| = |Z(H)| + \sum_{h \in H, h \notin Z(H)} |\mathrm{Orb}(h)|.$$

We have $p||H|$ and $|\mathrm{Orb}(h)| = [G : \mathrm{Stab}(h)]$ by Orbit-Stabilizer. Thus $p||\mathrm{Orb}(h)|$ for each $h \notin Z(H)$. We conclude that p divides $|Z(G)| = |H| - \sum_{h \in H, h \notin Z(H)} |\mathrm{Orb}(h)|$. Note $|Z(H)| > 1$ since the identity of H is contained in the center. Thus $|Z(H)| \geqslant p$ so H has a nontrivial center.

Fall 2017 Problem 1. Let G be a finite group, p a prime number, and S a Sylow p-subgroup of G. Let $N = \{g \in G | gSg^{-1} = S\}$. Let X and Y be two subsets of $Z(S)$ (the center of S) such that there is $g \in G$ with $gXg^{-1} = Y$. Show that there exists $n \in N$ such that $nxn^{-1} = gxg^{-1}$ for all $x \in X$.

Let G act on a set X with $g \cdot x = y$ for $g \in G$ and $x, y \in X$. We want to show that $\mathrm{Stab}(Y) = g\mathrm{Stab}(x)g^{-1} \subset G$. Let $h \in \mathrm{Stab}(y)$. Then $g^{-1}hg \cdot x = g^{-1}h \cdot y = g^{-1} \cdot y = x$ so $g^{-1}hg \in \mathrm{Stab}(x)$. We have $g^{-1}\mathrm{Stab}(y)g \subset \mathrm{Stab}(x)$. Next let $k \in \mathrm{Stab}(x)$. Then $gkg^{-1} \cdot y = gk \cdot x = g \cdot x = y$ and $g\mathrm{Stab}(x)g^{-1} \subset \mathrm{Stab}(y)$. Since conjugation by an element of a group is an invertible operation, $\mathrm{Stab}(y) = g\mathrm{Stab}(x)g^{-1}$.

We can define an N-action on S via conjugation. Define $\mathrm{Stab}(X) := \bigcap_{x \in X} \mathrm{Stab}(x) \subset G$. Since $X, Y \subset Z(S)$, we have $S \subset \mathrm{Stab}(X)$ and $S \subset \mathrm{Stab}(Y)$. Note that S is a Sylow p-subgroup of $\mathrm{Stab}(X)$ and $\mathrm{Stab}(Y)$. By the result above applied to each $y \in Y$, we have $\mathrm{Stab}(Y) = g\mathrm{Stab}(X)g^{-1}$. Conjugation preserves the order of subgroups so $gSg^{-1} \subset \mathrm{Stab}(Y)$ is a Sylow p-subgroup of $\mathrm{Stab}(Y)$. By Sylow's Second Theorem, the two Sylow p-subgroups S and gSg^{-1} are conjugate in $\mathrm{Stab}(Y)$. Thus there exists an $h \in \mathrm{Stab}(Y)$ such that $h(gSg^{-1})h^{-1} = S$. We note that $hg \in N$. Additionally, $(hg) \cdot x = h \cdot (gxg^{-1}) = gxg^{-1}$ since $h \in \mathrm{Stab}(Y)$. Let $n := hg \in N$ and $nxn^{-1} = gxg^{-1}$ for all $x \in X$.

Spring 2018 Problem 9. Show that there is no simple group of order 616.

As in Spring 2015 Problem 8, conjugation of a Sylow p-subgroup by an element $g \in G$ is another Sylow p-subgroup. If there is only one Sylow p-subgroup, then the Sylow p-subgroup is normal in G.

Let G be a group with order $616 = 2^3 \cdot 7 \cdot 11$. By Sylow's Third Theorem, the number of Sylow 11-subgroups m_{11} divides 56 and is congruent to 1 modulo 11. Thus we could have $m_{11} = 1$ or $m_{11} = 56$. As we will show, $m_{11} = 1$ implies the Sylow 11-subgroup is normal in G. Thus, assume $m_{11} = 56$. Next, the number of Sylow 7-subgroups m_7 divides 88 and is congruent to 1 modulo 7. We could have $m_7 = 1, 8, 22, 88$. The argument will work for larger choices for m_7 so assume $m_7 = 8$. The intersection of a Sylow 7-subgroup and Sylow 11-subgroup must be trivial by an order consideration. Thus the Sylow subgroups chosen account for $(11 + 55(10)) + (8(6)) = 609$ elements. A Sylow 2-subgroup of G will have order 8. As a result, there can be at most one Sylow 2-subgroup. Sylow's Theorems imply the existence of a Sylow 2-subgroup so $m_j = 1$ for some $j \in \{2, 7, 11\}$. By the above argument, we conclude that G has a normal subgroup and G is not simple.

Fall 2020 Problem 1. Let $p < q < r$ be primes and G a group of order pqr. Prove that G is not simple and, in fact, has a normal Sylow r-group.

We will first prove that G is not simple. Let n_p be the number of distinct Sylow p-subgroups, n_q be the number of distinct Sylow q-subgroups, and n_r be the number of distinct Sylow r-subgroups. By Sylow's Third Theorem, we know the following

$$n_p \equiv 1 \pmod{p}, \; n_p | qr$$
$$n_q \equiv 1 \pmod{q}, \; n_q | pr$$
$$n_r \equiv 1 \pmod{r}, \; n_r | pq.$$

We conclude that $n_r = 1, p, q, pq$. Since $r > p$ and $r > q$, p and q can't be congruent to 1 modulo r. Thus $n_r = 1$ or $n_r = pq$. If $n_r = 1$, we're done so assume $n_r = pq$. Every Sylow r-subgroup contains the identity and $r - 1$ order r elements of G. Thus there are $pq(r-1) = pqr - pq$ order r elements of G. Similarly, $n_q = 1, p, r, pr$. Since $q > p$,

p can't be congruent to 1 modulo q. If $n_q = 1$, we're done so assume that $n_q = r$, the smallest other possibility. As above, there are $r(q-1) = rq - r$ elements of order q in G. We have $n_p = 1, q, r, qr$ so assume that $n_p = q$. Then there are $q(p-1) = pq - q$ elements of order p in G. In total this accounts for

$$(pqr - pq) + (rq - r) + (pq - q) + 1 = pqr + rq - r - q + 1$$

elements of G. Since r and q are greater than 1, $rq \geqslant r + q$ and this exceeds the order of G. Thus there is some normal Sylow subgroup and G is not simple.

Let N be a normal Sylow subgroup of G. If $|N| = r$, we are done so assume $|N| = q$ without loss of generality. Then G/N is a group of order pr, which implies that G/N has a normal subgroup of order r. By the subgroup correspondence, there is a normal subgroup H of G containing N for which H/N is order r. Thus $|H| = qr$ and H contains a normal subgroup of order r denoted P_r. We want to prove that P_r is normal in G. Let $g \in G$. Then $|gP_rg^{-1}| = r$ and $gP_rg^{-1} \subset H$ since H is normal in G. Since P_r is a normal Sylow r-subgroup of H, P_r is the unique Sylow r-subgroup of H. We conclude that $gP_rg^{-1} = P_r$ and P_r is normal in G.

Fall 2020 Problem 2. Show that groups of order $231 = (3)(7)(11)$ are semi-direct products and show that there are exactly two such groups up to isomorphism.

Let G be a group of order 231 with P_3 a Sylow 3-subgroup, P_7 a Sylow 7-subgroup, and P_{11} a Sylow 11-subgroup. Since $|P_i \cap P_j| = 1$ for distinct i and j in $\{3, 7, 11\}$, we conclude that $|G| = |P_3P_7P_{11}|$ and $G = P_3P_7P_{11}$. By Fall 2020 Problem 1, P_{11} is normal in G. Let n_7 be the number of distinct Sylow 7-subgroups in G. Sylow's Third Theorem proves that $n_7 \equiv 1 \pmod 7$ and $n_7|33$. The only option is $n_7 = 1$ and P_7 is normal in G. Thus the cyclic subgroup P_7P_{11} of order 77 is normal in G and $G \simeq P_7P_{11} \rtimes_\varphi P_3$. We have $\mathrm{Aut}(P_7P_{11}) \simeq \mathbb{Z}/6\mathbb{Z} \times \mathbb{Z}/10\mathbb{Z}$ and P_3 cyclic of order 3. Therefore, $\varphi : P_3 \to \mathrm{Aut}(P_7P_{11})$ is either trivial or sends a generator of P_3 to an order 3 element of $\mathbb{Z}/6\mathbb{Z}$. The cases of the latter produce isomorphic semi-direct products so there are only two groups of order 231 up to isomorphism.

Math 210A Discussion Week 5

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Definition 1. Let G be a group. A *representation* of G is a homomorphism $\rho : G \to \mathrm{GL}(V)$ for some vector space V over a field F. For finite groups G, we will denote by $\dim(\rho)$ the dimension of V as an F-vector space.

Definition 2. Let $\rho : G \to \mathrm{GL}(V)$ be a representation of a group G. A subspace W of V is G-invariant if $\rho(g)w \in W$ for each $g \in G$. An *irreducible* representation $\rho : G \to \mathrm{GL}(V)$ is one in which there is no non-trivial, proper G-invariant subspace of V.

Definition 3. Let G be a group and F a field. We can define the *group algebra*, denoted $F[G]$, as the free vector space over F generated by the set G together with multiplication induced by the group law.

Example 1. Let G be cyclic of order n generated by $g \in G$. Then every element of $F[G]$ can be written as $\sum_{i=0}^{n-1} a_i g^i$ where $a_i \in F$. The multiplication works as follows

$$g \cdot \left(\sum_{i=0}^{n-1} a_i g^i \right) = a_{n-1}e + \sum_{i=1}^{n-1} a_{i-1} g^i$$

where e is the identity element of G.

A representation $\rho : G \to \mathrm{GL}(V)$ gives V the structure of an $F[G]$-module. An $F[G]$-module V defines a representation $\rho : G \to \mathrm{GL}(V)$ based on the action of each $g \in G$ on V. Further, isomorphic $F[G]$-modules correspond to isomorphic representations, or representations that differ by a base change. Thus the two languages are equivalent, and we will use the $F[G]$-module interpretation to find some nice properties of representations. For the qualifying exam, we will almost always take $F = \mathbb{C}$ and G finite.

Since $\mathbb{C}$ is characteristic 0 and algebraically closed, Artin-Wedderburn theorem implies that $\mathbb{C}[G]$ is a semi-simple F-algebra. More concretely, this means that

$$\mathbb{C}[G] \simeq \prod_{i=1}^{k} M_{d_i}(\mathbb{C})$$

where $M_{d_i}(\mathbb{C})$ is the algebra of $d_i \times d_i$ matrices over $\mathbb{C}$ and visa versa. Each component $M_{d_i}(\mathbb{C})$ defines an irreducible representation ρ_i of dimension d_i over $\mathbb{C}$. We have $\dim_{\mathbb{C}}(\mathbb{C}[G]) = |G|$ and $\dim_{\mathbb{C}}(M_{d_i}(G)) = d_i^2$ so

$$|G| = \sum_{i=1}^{k} d_i^2.$$

In other words, the sum of squares of the dimensions of irreducible representations of G equals the order of G.

The center of the group algebra, $Z(\mathbb{C}[G])$, is all elements $\alpha \in \mathbb{C}[G]$ that commute with each basis element $g \in \mathbb{C}[G]$. If $\alpha = \sum_{g \in G} a_g g$, we can show that $\alpha \in Z(\mathbb{C}[G])$ if and only if $a_g = a_{g'}$ whenever g and g' are in the same conjugacy class. Let $C_1, \ldots, C_k$ be distinct conjugacy classes of G. Then $\{u_1, \ldots, u_k\}$ is a basis for $Z(\mathbb{C}[G])$ where $u_i = \sum_{g \in C_i} g$. We conclude that $\dim(Z(\mathbb{C}[G])) = \#(\text{conjugacy classes of } G)$. On the other hand, $Z(M_{d_i}(\mathbb{C})) = \mathbb{C} \cdot I_{d_i}$ is one-dimensional so the number of irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes of G.

Example 2. Let G be a finite group. Two representations $\rho : G \to \mathrm{GL}_n(\mathbb{C})$ and $\mu : G \to \mathrm{GL}_n(\mathbb{C})$ are isomorphic if and only if $\rho(g) = P\mu(g)P^{-1}$ for $P \in \mathrm{GL}_n(\mathbb{C})$. Let $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ and $\mu : G \to \mathrm{GL}_1(\mathbb{C})$ be one-dimensional representations. Then $\mathrm{GL}(\mathbb{C}) \simeq \mathbb{C}^\times$ and ρ is isomorphic to μ if and only if $\rho = \mu$.

Example 3. Let A be a finite abelian group of order n. Then A has n distinct conjugacy classes, which implies n non-isomorphic irreducible representations. The dimensions d_i of the irreducible representations satisfy $n = \sum_{i=1}^n d_i^2$ so $d_i = 1$ for each $1 \leqslant i \leqslant n$.

Example 4. Let G be a finite group and $\rho : G \to \mathrm{GL}_1(\mathbb{C})$ a one-dimensional irreducible representation. Then $\mathrm{GL}_1(\mathbb{C}) \simeq \mathbb{C}^\times$ is abelian so ρ factors through $G/[G,G]$. We conclude that the number of one-dimensional irreducible representations of G is equal to the order of $G/[G,G]$.

Definition 4. Let $\rho : G \to \mathrm{GL}(V)$ be a representation. The *character* of ρ is defined as $\chi_\rho(g) = \mathrm{tr}(\rho(g))$.

Example 5. Let $\rho : G \to \mathrm{GL}(V)$ and $\mu : G \to \mathrm{GL}(W)$ be representations of a group G.

(1) If $\dim(\rho) = 1$, then $\chi_\rho(g) = \rho(g)$.

(2) If $\rho \simeq \mu$, then $\chi_\rho(g) = \chi_\mu(g)$ for all $g \in G$.

(3) $\chi_{\rho\oplus\mu}(g) = \chi_\rho(g) + \chi_\mu(g)$ for all $g \in G$.

(4) $\chi_\rho(hgh^{-1}) = \chi_\rho(g)$ for all $g, h \in G$.

(5) For $e \in G$ the identity, $\chi_\rho(e) = \dim(\rho)$.

Definition 5. Let $\mathrm{Ch}(G)$ be the vector space of functions $G \to F$ which are constant on conjugacy classes. Note that the characters of a group G are elements of this vector space. We can define a bilinear form on $\mathrm{Ch}(G)$:

$$B(\chi_\rho, \chi_\mu) = \frac{1}{n} \sum_{g \in G} \chi_\rho(g^{-1}) \chi_\mu(g).$$

The characters χ_i corresponding to the irreducible representations ρ_i form an orthonormal basis for $\mathrm{Ch}(G)$ with respect to B.

Theorem. Let $\rho_1, \ldots, \rho_k$ be the irreducible representations of a finite group G over $\mathbb{C}$ with corresponding characters $\chi_1, \ldots, \chi_k$.

(1) Every finite-dimensional representation ρ is isomorphic to $\bigoplus_{i=1}^k \rho_i^{B(\chi_\rho, \chi_i)}$.

(2) Two representations ρ and μ are isomorphic if and only if $\chi_\rho = \chi_\mu$.

(3) A representation ρ is irreducible if and only if $B(\chi_\rho, \chi_\rho) = 1$.

Example 6. Let $\rho_1, \ldots, \rho_k$ be the irreducible representations of a finite group G with corresponding characters $\chi_1, \ldots, \chi_k$. Let $C_1, \ldots, C_k$ be the conjugacy classes of G. Then

$$\sum_{i=1}^k \overline{\chi_i(g_{j_1})} \chi_i(g_{j_2}) = 0$$

for $g_{j_\ell} \in C_{j_\ell}$ and $j_1 \neq j_2$.

Example 7. The regular representation of G is given by acting on the vector space $F[G]$ by left multiplication. The representation will be $\rho : G \to \mathrm{GL}_n(F)$ where $n = |G|$. Each element $\rho(g)$ is a permutation matrix. Let $\{g_1, \ldots, g_n\}$ be the group eleemnts of G which form a basis for $F[G]$ as an F-vector space. If g is not the identity, then it fixes no elements of the basis and $\chi_\rho(g) = \mathrm{tr}(\rho(g)) = 0$. Let the irreducible representations of G be $\rho_1, \ldots, \rho_k$ with characters $\chi_1, \ldots, \chi_k$. We find that the regular representation breaks down as $\rho = \bigoplus_{i=1}^k \rho_i^{d_i}$ and $\chi_\rho = \sum_{i=1}^k d_i \chi_i$. The regular representation can be helpful in coming up with higher dimensional representations for a group since each irreducible representation is a direct summand.

Most of the qual problems on representation theory will ask us to find the character table of a given group G. The *character table* for a group G is constructed as follows. Each row will represent the character of an irreducible representation, which we will denote $\chi_1, \ldots, \chi_k$. Each column will represent a conjugacy class of G, which we will denote $C_1, \ldots, C_k$. By above, the table will have the same number of rows and columns. The ith row, jth column entry of the table will be $\chi_i(g_j)$ for $g_j \in C_j$.

A typical approach to one of these problems includes the following steps.

(1) Find the conjugacy classes of G. The number of conjugacy classes is the number of irreducible representations.

(2) The order of $G/[G,G]$ is the number of one-dimensional irreducible representations.

(3) Let d_i denote the dimension of the irreducible representation ρ_i with character χ_i. The equation $|G| = \sum_{i=1}^k d_i^2$ along with the number of one-dimensional irreducible representations can sometimes help us determine the dimensions of other irreducible representations.

(4) The column corresponding to the conjugacy class of the identity will be populated with the dimensions of each irreducible representation.

(5) The trivial one-dimensional representation $\rho(g) = 1$ will provide a row of all 1s.

(6) The rows satisfy an orthogonality condition $\sum_{i=1}^k |C_i|\overline{\chi_{j_1}(g_i)}\chi_{j_2}(g_i) = 0$ for $j_1 \neq j_2$ and some representative $g_i \in C_i$. Further $\sum_{i=1}^k |C_i||\chi_j(g_i)|^2 = n$.

(7) The columns satisfy an orthogonality condition $\sum_{i=1} \overline{\chi_i(g_{j_1})}\chi_i(g_{j_2}) = 0$ for $j_1 \neq j_2$ and $g_{j_\ell} \in C_{j_\ell}$.

Fall 2015 Problem 7, Spring 2016 Problem 8. Show the symmetric group S_4 has exactly two isomorphism classes of irreducible complex representations of dimension 3. Compute the characters of these two representations. Find the full character table.

We will first show that the abelianization $S_4/[S_4, S_4]$ has order 2. The commutator subgroup $[S_4, S_4]$ is generated by elements $ghg^{-1}h^{-1} \in S_4$. Each $ghg^{-1}h^{-1}$ is an even permutation so $[S_4, S_4] \subset A_4$. The nonidentity elements of A_4 are of the form $(ij)(k\ell)$ or (ijk) for $1 \leqslant i, j, k, \ell \leqslant 4$. Without loss of generality, we will show $(123), (14)(23) \in [S_4, S_4]$. Notice $(23)(12)(23)(12) = (123) \in [S_4, S_4]$ and $(123)(234)(132)(243) = (14)(23) \in [S_4, S_4]$ as desired. Thus $[S_4, S_4] = A_4$ and $|S_4/[S_4, S_4]| = 2$.

Each one-dimensional representation of S_4 is a group homomorphism $\rho : S_4 \to \mathbb{C}^\times$. Since $\mathbb{C}^\times$ is an abelian group, ρ factors uniquely through the abelian group $S_4/[S_4, S_4]$. If two one-dimensional representations are equal on $S_4/[S_4, S_4]$, then they are equal as homomorphisms from S_4. Thus the number of one-dimensional representations of $S_4/[S_4, S_4]$ is equal to the number of one-dimensional representations of S_4. By above, $S_4/[S_4, S_4]$ has two conjugacy classes so it has two one-dimensional irreducible representations. We conclude that S_4 should have two one-dimensional representations. (This works for one-dimensional irreducible representations of any group.)

Now the trivial representation and the sign representation, $\mathrm{sgn} : S_4 \to \mathbb{C}^\times$, are the two one-dimensional representations of S_4. The conjugacy classes of S_4 are based on cycle type of which there are five. Since $|S_4| = 24$, we have $24 = 1 + 1 + a^2 + b^2 + c^2$ for $a, b, c \in \mathbb{N}$ representing the dimensions of the three other irreducible representations. If we take $c \geqslant 4$, we are left with $a^2 + b^2 = 6$, which cannot occur. Thus $1 < a, b, c \leqslant 3$. We cannot have $a = b = c = 2$ so, without loss of generality, take $c = 3$. Then we need $13 = a^2 + b^2$ so the only option is $a = 2$ and $b = 3$. Thus S_4 has two 3-dimensional irreducible representations.

We will now realize the two irreducible representations of dimension 3. Define the vector space

$$V := \left\{ (v_i) \in \mathbb{R}^4 : \sum_{i=1}^{4} v_i = 0 \right\}.$$

Then V has a left S_4 action via $\sigma(v_i) = (v_{\sigma(i)})$ for $\sigma \in S_4$ and $\{(-1, 1, 0, 0), (-1, 0, 1, 0), (-1, 0, 0, 1)\}$ is a basis for V. The action described gives an irreducible representation for S_4 since $(23)(-1, 1, 0, 0) = (-1, 0, 1, 0)$ and $(24)(-1, 1, 0, 0) = (-1, 0, 0, 1)$. In other words, there is no S_4-invariant subspace of V. Let $\rho : S_4 \to M_3(\mathbb{C})$ denote this 3-dimensional irreducible representation.

	e	(12)	(123)	(12)(34)	(1234)
χ_{sgn}	1	-1	1	1	-1
χ_ρ	3	1	0	-1	-1

Now $\rho \otimes \mathrm{sgn}$ is an irreducible representation of $S_4 \times S_4$. Include S_4 along the diagonal of $S_4 \times S_4$ to make $\rho \otimes \mathrm{sgn}$ a representation of S_4. The character $\chi_{\rho\otimes\mathrm{sgn}}(g) = \chi_\rho(g)\chi_{\mathrm{sgn}}(g)$ gives the following row of the character table.

	e	(12)	(123)	(12)(34)	(1234)
$\chi_{\rho\otimes\mathrm{sgn}}$	3	-1	0	-1	1

We have an inner product on the space of class functions such as $\langle \chi_\mu, \chi_\nu \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_\mu(g)\chi_\nu(g^{-1})$. We know that $\langle \chi_{\rho\otimes\mathrm{sgn}}, \chi_{\rho\otimes\mathrm{sgn}} \rangle = 1$ if and only if $\rho \otimes \mathrm{sgn}$ is an irreducible representation. We note that the number of elements in each conjugacy class are $1, 6, 8, 3, 6$ respectively. Since g^{-1} and g are in the same conjugacy class for all $g \in S_4$,

$$\langle \chi_{\rho\otimes\mathrm{sgn}}, \chi_{\rho\otimes\mathrm{sgn}} \rangle = \frac{1}{24}(1(9) + 6(1) + 8(0) + 3(1) + 6(1)) = 1.$$

Thus $\rho \otimes \mathrm{sgn}$ is the other irreducible representation of S_4.

The only remaining row of the character table corresponds to the 2-dimensional irreducible representation which we denote $\mu : S_4 \to M_2(\mathbb{C})$. We will use column orthogonality to complete the table below.

	e	(12)	(123)	(12)(34)	(1234)
χ_{trivial}	1	1	1	1	1
χ_{sgn}	1	-1	1	1	-1
χ_μ	2	0	-1	2	0
χ_ρ	3	1	0	-1	-1
$\chi_{\rho\otimes\mathrm{sgn}}$	3	-1	0	-1	1

Fall 2016 Problem 4. Let D be a dihedral group of order $2p$ with normal cyclic subgroup C of order p for an odd prime p. Find the number of n-dimensional irreducible representations of D (up to isomorphisms) over $\mathbb{C}$ for each n, and justify your answer.

Let $D := \langle r, s : r^p = s^2 = e, rs = sr^{-1}\rangle$ be the dihedral group of order $2p$. We will find the commutator subgroup $[D, D] \subset D$. Any element of the commutator subgroup is of the form $(r^i s)(r^j s)(r^i s)^{-1}(r^j s)^{-1}$ for some $0 \leqslant i, j \leqslant p-1$. Reducing this, we end up with r^{2i-2j}. Further, $r^{\frac{p+1}{2}} s r^{p-\frac{p+1}{2}} s^{-1} = r^{\frac{p+1}{2}} r^{\frac{p+1}{2}} ss = r^{\frac{2p+2}{2}} = r \in [D, D]$. Thus $[D, D]$ is the subgroup of D generated by r and $|D/[D, D]| = 2$. There are two non-isomorphic classes of one-dimensional representations of D.

We now classify the conjugacy classes of D_p. Note that it is sufficient to conjugate each element only by the generators r and s. The identity makes up one conjugacy class. When we conjugate s we notice $r^i s r^{p-i} = r^{2i}s$. Since p is odd, we can continue this process to obtain the conjugacy class $\{s, rs, \ldots, r^{p-1}s\}$. When we conjugate r^i we have $sr^i s^{-1} = sr^i s = r^{p-i}$ for $1 \leqslant i \leqslant p-1$. Conjugating by s again yields $sr^{p-i}s^{-1} = sr^{p-i}s = r^i$. Thus we have the conjugacy classes $\{r^i, r^{p-i}\}$ for $1 \leqslant i \leqslant \frac{p-1}{2}$. In total, this is $\frac{p+3}{2}$ conjugacy classes.

Using the intuition of D as permutations of vertices of a regular p-gon, we can construct the classes of 2-dimensional irreducible representations. Define the rotation by $\frac{2\pi k}{p}$ counterclockwise in the plane,

$$\phi_k(r) = \begin{pmatrix} \cos(2\pi k/p) & -\sin(2\pi k/p) \\ \sin(2\pi k/p) & \cos(2\pi k/p) \end{pmatrix}$$

and the reflection about the x-axis in the plane,

$$\phi_k(s) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

for $1 \leqslant k \leqslant \frac{p-1}{2}$. Each ϕ_k is an irreducible representation of D since there are no subspaces of $\mathbb{C}^2$ invariant under these transformations. Further, these are non-isomorphic irreducible representations since the characters $\chi_{\phi_k}(r) = 2\cos(2\pi k/p)$ differ for each k.

The sum of the squares of the dimensions of these representations is $1 + 1 + \left(\frac{p-1}{2}\right)2^2 = 2 + (2p-2) = 2p$, the order of the group. Thus these are all isomorphism classes of irreducible representations of D over $\mathbb{C}$. We conclude that there are two one-dimensional and $\frac{p-1}{2}$ two-dimensional isomorphism classes of irreducible complex representations of D.

Spring 2017 Problem 2. Let G be the group with presentation $\langle x, y : x^4 = 1, y^5 = 1, xyx^{-1} = y^2\rangle$, which has order 20. Find the character table of G.

We will first find the conjugacy classes of G. Note that we only need to check conjugation by the generators x and y. Since $xy = y^2x$, we can write each element of G as $y^i x^j$ for some $0 \leqslant i < 5$ and $0 \leqslant j < 4$. Additionally,

$$(y^i x^j)(y^k x^\ell)(y^i x^j)^{-1} = y^{i+2^j k} x^{j+\ell} x^{-j} y^{-i} = y^{i+2^j k} x^\ell y^{-i} = y^{-i+2^j k} x^\ell$$

so the exponent of x remains unchanged by conjugation. By the formula above, conjugating $y^k x^\ell$ by y will result in $y^{k-1}x^\ell$. Thus the conjugacy classes are

$$\{1\}, \{y, y^2, y^3, y^4\}\{x, yx, y^2x, y^3x, y^4x\}, \{x^2, yx^2, y^2x^2, y^3x^2, y^4x^2\}, \{x^3, yx^3, y^2x^3, y^3x^3, y^4x^3\},$$

which implies 5 isomorphism classes of irreducible representations. We will now find the commutator subgroup $[G, G]$. The generators of $[G, G]$ have the form $(y^i x^j)(y^k x^\ell)(y^i x^j)^{-1}(y^k x^\ell)^{-1} = (y^{-i+2^j k} x^\ell)x^{-\ell}y^{-k} = y^{-i+(2^j-1)k}$. We can pick $i = 4$, $j = 0$, $k = 0$, and $\ell = 1$, which implies $[G, G]$ is the cyclic subgroup of G generated by y. Then the number of isomorphism classes of one-dimensional representations is $|G/[G, G]| = 4$ by the argument in Fall 2015 Problem 7. There are 4 one-dimensional representations and 5 conjugacy classes. Since the order of G is the sum of the squares of the dimensions of the irreducible representations, $20 = 1^2 + 1^2 + 1^2 + 1^2 + k^2$ so $k = 4$.

We will now determine the 4 one-dimensional representations. Since x is order 4, it must map to $\pm 1, \pm i$ in $\mathbb{C}^\times$. Similarly, y is order 5 so y must map to a fifth root of unity in $\mathbb{C}^\times$. The character is equal to the representation in the one-dimensional case so the representation is the same on each conjugacy class. Let $\rho_i : G \to \mathbb{C}^\times$ be one-dimensional representations for $1 \leqslant i \leqslant 3$ and $\mu : G \to \mathrm{GL}_4(\mathbb{C})$ be the 4-dimensional irreducible representation.

For $\rho_i : G \to \mathbb{C}^\times$, $\rho_i(y) = \rho_i(y^2) = \rho_i(y)^2$ so $\rho_i(y) = 1$. We can fill in the character table below based on the image of x. The last row of the table is found by column orthogonality.

	1	y	x	x^2	x^3
χ_{trivial}	1	1	1	1	1
χ_{ρ_1}	1	1	i	-1	$-i$
χ_{ρ_2}	1	1	-1	1	-1
χ_{ρ_3}	1	1	$-i$	-1	i
χ_μ	4	-1	0	0	0

Spring 2018 Problem 6. Let G be a group with a normal subgroup $N = \langle y, z\rangle$ isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$. Suppose that G has a subgroup $Q = \langle x\rangle$ isomorphic to the cyclic group $\mathbb{Z}/3\mathbb{Z}$ such that the composition $Q \subset G \to G/N$ is an isomorphism. Finally, suppose that $xyx^{-1} = z$ and $xzx^{-1} = yz$. Compute the character table of G.

We will find the conjugacy classes of G. Since $xy = zx$ and $xz = yzx$, we can write every element of G as $y^i z^j x^k$ for $0 \leqslant i, j \leqslant 1$ and $0 \leqslant i \leqslant 2$. The relations allow reduction to the form $y^i z^j x^k$ without changing the x exponent. As a result, conjugation by any element will preserve the x exponent of any element. We will show that the conjugacy classes are based on the exponent of x. The relations of G produce the conjugacy class $\{y, z, yz\}$. In the equations below, we start with x.

$$\begin{aligned} yxy^{-1} &= yxy = zx \\ y(zx)y^{-1} &= yz^2x = yx \\ z(zx)z^{-1} &= xz = yzx \end{aligned}$$

A similar argument starting with x^2 gives the conjugacy class breakdown below.

$$\{e\}, \{y, z, yz\}, \{x, yx, zx, yzx\}, \{x^2, yx^2, zx^2, yzx^2\}$$

Note that $|G| = 12$. Thus the sum of 1 and three squares needs to be $|G| = 12$. We cannot have an irreducible representations of dimension higher than three. The only option is $12 = 1^2 + 1^2 + 1^2 + 3^2$ so there should be three isomorphism classes of one-dimensional representations and one isomorphism class of 3-dimensional irreducible representations.

We will first classify the characters of the one-dimensional irreducible representations. Let $\rho_i : G \to \mathbb{C}^\times$ for $1 \leqslant i \leqslant 3$ be the one-dimensional representations. Since y and z are order 2 elements of G, they must map to ± 1 in $\mathbb{C}^\times$. Similarly, x will be sent to a third root of unity. The group $\mathbb{C}^\times$ is abelian so $\rho(z) = \rho(xyx^{-1}) = \rho(x)\rho(y)\rho(x)^{-1} = \rho(y)$ and $\rho(yz) = \rho(xzx^{-1}) = \rho(x)\rho(z)\rho(x)^{-1} = \rho(z)$. Let ξ be a primitive third root of unity. We find the final row of the character table by column orthogonality and the identity $\sum_{i=1}^{3} \xi^i = 0$.

	1	y	x	x^2
χ_{trivial}	1	1	1	1
χ_{ρ_1}	1	1	ξ	ξ^2
χ_{ρ_2}	1	1	ξ^2	ξ
χ_μ	3	-1	0	0

Fall 2018 Problem 11. Let G be a finite group, ω be a primitive 3rd root of 1 in $\mathbb{C}$ and suppose that the complex character table of G contains the row

$$1 \quad \omega \quad \omega^2 \quad 1.$$

Determine the whole complex character table of G, the order of the group and the order of its conjugacy classes.

Note that the number of columns, four, determines the number of conjugacy classes of G and the number of isomorphism classes of irreducible representations. The first row of the character table corresponds to the trivial representation. Let $\rho : G \to \mathbb{C}$ be the one-dimensional representation described in the row given. Then we can construct a one-dimensional representation $\rho \otimes \rho : G \times G \to \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \simeq \mathbb{C}$. By including G in $G \times G$ via the diagonal homomorphism, we find $\rho \otimes \rho$ describes a one-dimensional representation with $\chi_{\rho\otimes\rho}(g) = \chi_\rho(g)^2$. Since the characters $\chi_{\rho\otimes\rho}$ differ from the current rows, $\rho\otimes\rho$ describes a distinct isomorphism class of one-dimensional representations.

By orthogonality of the second/third column and the first column, we find the zeros in the fourth row. Let $a := \chi_\mu(e)$ and $b := \chi_\mu(g)$ for $g \in C_4$. Then $ab = -3$ by the orthogonality of columns one and four. Since a represents the dimension of the irreducible representation $\mu : G \to M_a(\mathbb{C})$, $a > 0$ is an integer so $b \in \mathbb{Q}$. With $|G|$ finite, the trace of $\mu(g)$ is the sum of eigenvalues that are all roots of unity. Thus $b \in \mathbb{Q}$ is an algebraic integer so $b \in \mathbb{Z}$. We conclude that $a = 1$ and $b = -3$ or $a = 3$ and $b = -1$. If $a = 1$, then $|G| = 4$. The order of some $g \in C_2$ must be divisible by 3 since $\rho(g^3) = \rho(g)^3 = 1$. This contradicts the order of G so $a \neq 1$. Thus $a = 3$ and $b = -1$.

As a result, $|G| = 1^2 + 1^2 + 1^2 + 3^2 = 12$. The rows are orthonormal under the inner product $\langle v, w\rangle = \frac{1}{|G|}\sum_{i=1}^4 |C_i| v_i \overline{w_i}$. Row three implies $1 = \frac{9+|C_4|}{12}$ and $|C_4| = 3$. The inner product of rows two and one gives $0 = \frac{1+|C_2|\omega+|C_3|\omega^2+3}{12}$. Similarly, the inner product of rows three and one gives $0 = \frac{1+|C_2|\omega^2+|C_3|\omega+3}{12}$. Thus $|C_2| = |C_3|$ with 8 elements between the two conjugacy classes. We conclude $|C_2| = |C_3| = 4$.

	$C_1 = \{e\}$	C_2	C_3	C_4
χ_{trivial}	1	1	1	1
χ_ρ	1	ω	ω^2	1
$\chi_{\rho\otimes\rho}$	1	ω^2	ω	1
χ_μ	3	0	0	-1

Fall 2019 Problem 7. Let G be the group of order 12 with presentation

$$G = \langle g, h : g^4 = 1, h^3 = 1, ghg^{-1} = h^2\rangle.$$

Find the conjugacy classes of G and the values of the characters of the irreducible complex representations of G of dimension greater than 1 on representatives of these classes.

The final relation of G implies that $gh = h^2g$ and $gh^2 = hg$. We can use these relations to write every element of G as g^ih^j for $0 \leqslant i \leqslant 3$ and $0 \leqslant j \leqslant 2$. Further, we have the relations $h^2g^3 = g^3h$ and $hg^3 = g^3h^2$ by inverting the above relations. Clearly, $C_1 = \{e\}$ is a conjugacy class. The relations

$$\begin{aligned} ghg^{-1} &= ghg^3 = h^2 \\ gh^2g^{-1} &= gh^2g^3 = h \end{aligned}$$

show that $C_2 = \{h, h^2\}$ is a conjugacy class. We find

$$\begin{aligned} hgh^{-1} &= hgh^2 = gh \\ h(gh)h^{-1} &= gh^2 \\ g(gh)g^{-1} &= g^2hg^3 = gh^2 \\ h(gh^2)h^{-1} &= hgh = g \\ g(gh^2)g^{-1} &= g^2h^2g^3 = gh \end{aligned}$$

so $C_3 = \{g, gh, gh^2\}$ is a conjugacy class. By similar computation, we have conjugacy class $C_4 = \{g^3, g^3h, g^3h^2\}$. The equations

$$\begin{aligned} hg^2h^{-1} &= hg^2h^2 = gh^2gh^2 = g^2 \\ h(g^2h)h^{-1} &= hg^2 = gh^2g = g^2h \\ g(g^2h)g^{-1} &= g^3hg^3 = g^2h^2 \\ h(g^2h^2)h^{-1} &= hg^2h = gh^2gh = g^2h^2 \\ g(g^2h^2)g^{-1} &= g^3h^2g^3 = g^2h \end{aligned}$$

prove that $C_5 = \{g^2\}$ and $C_6 = \{g^2h, g^2h^2\}$ are conjugacy classes. All elements of G have been placed in conjugacy classes.

The commutator $[G, G]$ has elements of the form $ghg^{-1}h^{-1} = ghg^3h^2 = h$. Thus $\langle h\rangle \subset [G, G]$. We see that $G/\langle h\rangle$ is cyclic of order 4 and, thus, abelian. We conclude $[G, G] = \langle h\rangle$ and there are $|G/[G, G]| = 4$ one-dimensional non-isomorphic irreducible representations of G. Each one-dimensional $\rho_i : G \to \mathbb{C}^\times$ sends h to 1. The image of g must be a fourth root of unity. Further, $12 = 4 + a^2 + b^2$ for a and b the dimensions of the other irreducible representations of G. We see that $a < 3$ and $b < 3$ so $a = b = 2$ so we obtain the following character table.

	e	h	g	g^2	g^3	g^2h
χ_1	1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$	-1
χ_3	1	1	-1	1	-1	1
χ_4	1	1	$-i$	-1	i	-1
χ_5	2					
χ_6	2					

We will construct a two-dimensional irreducible representation of G over $\mathbb{C}$. Define a set map μ on the generators

$$\mu(g) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$
$$\mu(h) = \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}}. \end{pmatrix}$$

Then the image of g has order 4 in $\mathrm{GL}_2(\mathbb{C})$ and the image of h has order 3 in $\mathrm{GL}_2(\mathbb{C})$. Further,

$$\begin{aligned} \mu(ghg^{-1}) &= \mu(g)\mu(h)\mu(g)^{-1} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} e^{\frac{2\pi i}{3}} & 0 \\ 0 & e^{\frac{4\pi i}{3}} \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} e^{\frac{4\pi i}{3}} & 0 \\ 0 & e^{\frac{2\pi i}{3}} \end{pmatrix} \\ &= \mu(h)^{-1} \end{aligned}$$

so $\mu : G \to \mathrm{GL}_2(G)$ is a group homomorphism as desired. There is no non-trivial, proper G-invariant subspace of $\mathbb{C}^2$ which proves μ is irreducible. Compute the characters χ_5 by taking the traces of the relevant matrices. We can complete the final row of the character table by column orthogonality of column j with column 1.

	e	h	g	g^2	g^3	g^2h
χ_1	1	1	1	1	1	1
χ_2	1	1	i	-1	$-i$	-1
χ_3	1	1	-1	1	-1	1
χ_4	1	1	$-i$	-1	i	-1
χ_5	2	-1	0	-2	0	1
χ_6	2	-1	0	2	0	-1

Math 210A Discussion Week 7

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Definition 1. A *category* $\mathcal{C}$ consists of a collection of objects $\mathrm{Ob}(\mathcal{C})$, a collection of morphisms between objects $\mathrm{Hom}_{\mathcal{C}}(X, Y)$, and a composition operation $\circ$ on morphisms such that

(i) $h \circ (g \circ f) = (h \circ g) \circ f$

(ii) for each object X of $\mathcal{C}$ there is a unique morphism id_X that satisfies $f = f \circ \mathrm{id}_X$ and $g = \mathrm{id}_X \circ g$ for morphisms into or out of X.

Definition 2. A *small* category is one in which the objects and morphisms form a set. A *locally small* category $\mathcal{C}$ is one in which $\mathrm{Hom}_{\mathcal{C}}(X, Y)$ is a set for any two objects X and Y. Note that the collection of all morphisms in the category might be too large to form a set.

Example 1. The most useful examples of categories for the qual are:

1. The category of sets (Sets)
2. The category of groups (Grps) with the full subcategory of abelian groups (Ab)
3. The category of rings (Rings) with the full subcategory of commutative rings (CRings)
4. The category of R-modules over a ring R (R-Mod)

For each term we define, we want to know the technical construction in each of the above categories. Once we're familiar with each category, we will hopefully be able to come up with simple counterexamples to qual problems.

Definition 3. Let X_1 and X_2 be objects of a category $\mathcal{C}$. The *product* $X_1 \times X_2$ is equipped with morphisms $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ and satisfies the following universal property. Given an object Y and two morphisms $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$, there is a unique morphism $f : Y \to X_1 \times X_2$ such that $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

$$\begin{tikzcd}
Y \arrow[dr, dashed, "f"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\
& X_1 \times X_2 \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X_2 \\
& X_1 &
\end{tikzcd}$$

We can extend this definition to a product on infinitely many objects of $\mathcal{C}$ which will be denoted $\prod_i X_i$. Dual to the notion of a product is the coproduct.

Definition 4. Let X_1 and X_2 be objects of a category $\mathcal{C}$. The *coproduct* $X_1 \coprod X_2$ is equipped with morphisms $\iota_1 : X_1 \to X_1 \times X_2$ and $\iota_2 : X_2 \to X_1 \times X_2$ and satisfies the following universal property. Given an object Y and two morphisms $f_1 : X_1 \to Y$ and $f_2 : X_2 \to Y$, there is a unique morphism $f : X_1 \coprod X_2 \to Y$ such that $f_1 = f \circ \iota_1$ and $f_2 = f \circ \iota_2$.

$$\begin{tikzcd}
Y & & \\
& X_1 \coprod X_2 \arrow[ul, dashed, "f"'] & X_2 \arrow[l, "\iota_2"] \arrow[ull, bend right, "f_2"'] \\
& X_1 \arrow[u, "\iota_1"] \arrow[uul, bend left, "f_1"] &
\end{tikzcd}$$

We can likewise extend the definition of coproducts to infinite families of objects.

Example 2.

1. In Sets, the product is the cartesian product and the coproduct is disjoint union.
2. In Grps, the product is the direct product and the coproduct is the free product. In Ab, the product is direct product while the coproduct is direct sum. For finite families, the product and coproduct coincide in Ab.
3. In Rings, the product is direct product and the coproduct is similar to the free product on groups. In CRings, the product is direct product and finite coproducts are given by tensoring over $\mathbb{Z}$.
4. Finite products and coproducts are isomorphic and given by the direct sum in R-Mod.

Via the universal properties of products and coproducts, we can construct the following bijections

$$\mathrm{Hom}_{\mathcal{C}}\left(X, \prod_i Y_i\right) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(X, Y_i)$$
$$\mathrm{Hom}_{\mathcal{C}}\left(\coprod_i X_i, Y\right) \simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(X_i, Y).$$

Spring 2015 Problem 1. What are the coproducts in the category of groups?

We will define the free product of a family of groups $G_{i i \in I}$. As a set, $*_{i\in I} G_i$ is all words on the letters $\bigcup_{i\in I} G_i$. We reduce letters from the same group via the group multiplication. Define the group operation as concatenation. The identity element is the empty word, concatenation is associative, and the inverse of a reduced word $g_1 \cdots g_n$ is $g_n^{-1} \cdots g_1^{-1}$. Thus the free product of a family of groups is a group.

Define the inclusion homomorphisms $i_j : G_j \to *_{k\in I} G_k$ as $i_j(g) = g$. We want to show that $*_{i\in I} G_i$ satisfies the universal property of the coproduct. Let $f_i : G_i \to A$ be a family of group homomorphisms. For the diagram below to commute, $h : *_{k\in I} G_k \to A$ must be defined as $h(g) = f_j(g)$ for $g \in G_j$. Then we extend h to a group homomorphism. For a reduced word $g_1 \cdots g_n \in *_{k\in I} G_k$, we have $h(g_1 \cdots g_n) = h(g_1) \cdots h(g_n) = f_{j_1}(g_1) \cdots f(g_n)$ for $g_i \in G_{j_i}$. Since h is uniquely determined by the $\{f_j\}_{j\in I}$, the free product is the coproduct in the category of groups.

$$\begin{tikzcd}
 & G_{j_2} \arrow[d, "i_{j_2}"] \arrow[dr, bend left, "f_{j_2}"] & \\
G_{j_1} \arrow[r, "i_{j_1}"] \arrow[rr, bend right, "f_{j_1}"'] & *_{k\in I} G_k \arrow[r, dashed, "h"] & A
\end{tikzcd}$$

Fall 2018 Problem 8. Give an example of a diagram of commutative rings whose colimit in the category of commutative rings is different from its colimit in the larger category of rings (and ring homomorphisms).

We will show that the coproduct of two commutative rings is the tensor product over $\mathbb{Z}$. Let A, B, C be commutative rings with ring homomorphisms $f : A \to C$ and $g : B \to C$. We need $h(i_A(a)) = h(a \otimes 1) = f(a)$ and $h(i_B(b)) = h(1 \otimes b) = g(b)$ for $a \in A$ and $b \in B$. Extend h to a commutative ring morphism so $h(a \otimes b) = f(a)g(b)$ for $a \otimes b \in A \otimes_{\mathbb{Z}} B$. Thus h is the unique commutative ring morphism that causes the diagram to commute.

$$\begin{tikzcd}
 & B \arrow[d, "i_B"] \arrow[dr, bend left, "g"] & \\
A \arrow[r, "i_A"] \arrow[rr, bend right, "f"'] & A \otimes_{\mathbb{Z}} B \arrow[r, dashed, "h"] & C
\end{tikzcd}$$

We will now show that the tensor product over $\mathbb{Z}$ is not the coproduct in the category of rings. Let $A = B = C = M_2(\mathbb{Q})$ and take $f = g = \mathrm{id}_{M_2(\mathbb{Q})}$. Then $h : M_2(\mathbb{Q}) \otimes_{\mathbb{Z}} M_2(\mathbb{Q}) \to M_2(\mathbb{Q})$ can be defined as $h(a \otimes b) = ab$ or $h(a \otimes b) = ba$. These two ring morphisms are not equal since $M_2(\mathbb{Q})$ is not commutative. Thus $M_2(\mathbb{Q}) \otimes_{\mathbb{Z}} M_2(\mathbb{Q})$ does not satisfy the universal property of the coproduct.

Definition 5. A *covariant functor* $F : \mathcal{C} \to \mathcal{D}$ for categories $\mathcal{C}$ and $\mathcal{D}$ is a collection of functions $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ and $\mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ such that

(i) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$

(ii) $F(g \circ f) = F(g) \circ F(f)$.

A *contravariant functor* satisfies $F(g \circ f) = F(f) \circ F(g)$.

Definition 6. Let F and G be functors $\mathcal{C}$ to $\mathcal{D}$. A natural transformation $\alpha : F \to G$ is a collection of morphisms $\alpha_X : F(X) \to G(X)$ such that for $f : X \to Y$ in $\mathcal{C}$, the following diagram commutes.

$$\begin{array}{ccc} F(X) & \xrightarrow{\alpha_X} & G(X) \\ \big\downarrow{\scriptstyle F(f)} & & \big\downarrow{\scriptstyle G(f)} \\ F(Y) & \xrightarrow{\alpha_Y} & G(Y) \end{array}$$

Natural transformations are mappings between functors that preserve the structure of the underlying categories. If we look at the category of functors between $\mathcal{C}$ and $\mathcal{D}$, the morphisms would be natural transformations. A natural isomorphism $\alpha : F \to G$ would be an isomorphism on each object X of $\mathcal{C}$.

Example 3. Let X be an object of a locally small category $\mathcal{C}$. Define the functor $R^X : \mathcal{C} \to \mathrm{Sets}$ as

$$R^X(Y) = \mathrm{Hom}_{\mathcal{C}}(X, Y)$$
$$R^X(f)(g) = f \circ g$$

for $f : Y \to Z$ and $g \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. A functor F is *represented by* X if F is naturally isomorphic to R^X.

Lemma 1 (Yoneda)**.** Let $\mathcal{C}$ be a locally small category and fix an object X of $\mathcal{C}$. Let $F : \mathcal{C} \to \mathrm{Sets}$ be a functor. There is a bijection $\varphi : \mathrm{Nat}(R^X, F) \to F(X)$ given by $\varphi(\alpha) = \alpha(\mathrm{id}_X)$.

Yoneda Lemma is possibly the most important result in this section. It tells us that instead of studying locally small categories, it might be helpful to embed them into the category of functors into Sets. We have some understanding of the category of Sets that might provide intution about the category $\mathcal{C}$. As a corollary to Yoneda Lemma, we obtain an isomorphism

$$\mathrm{Nat}(R^X, R^Y) \simeq \mathrm{Hom}(Y, X).$$

There is a deep relationship between functors represented by X and the object X. One way in which we will use Yoneda Lemma is the situation where $\mathrm{Hom}_{\mathcal{C}}(A, B) \simeq \mathrm{Hom}_{\mathcal{C}}(C, B)$ for all objects B. Yoneda Lemma tells us that $C \simeq A$.

Definition 7. Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be two functors. We say that F and G form an *adjunction pair* with F a *left adjoint* to G and G a *right adjoint* to F if

$$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, G(Y))$$

for all $X \in \mathrm{Ob}(\mathcal{C})$ and $Y \in \mathrm{Ob}(\mathcal{D})$ such that the family of bijections is natural in X and Y.

Alternatively, F and G form an *adjunction pair* with F a *left adjoint* to G and G a *right adjoint* to F if there are natural transformations $\varepsilon : FG \to 1_{\mathcal{C}}$ and $\eta : 1_{\mathcal{D}} \to GF$ such that

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations on F and G respectively. We call ε the *counit* and η the *unit*.

Spring 2015 Problem 2. Let $\mathcal{C}$ be the category of groups and $\mathcal{C}'$ be its full subcategory with objects the abelian groups. Let $F : \mathcal{C}' \to \mathcal{C}$ be the inclusion functor. Determine the left adjoint of F and show that F has no right adjoint.

Let $f : G \to H$ be a group homomorphism where H is abelian. The commutator subgroup $[G, G]$ is generated the subgroup generated by $\{g_1g_2g_1^{-1}g_2^{-1} \in G\, g_1, g_2 \in G\}$. For $g_1, g_2 \in G$, we have $(g_1[G, G])(g_2[G, G]) = g_1g_2[G, G] = g_1g_2(g_2^{-1}g_1^{-1}g_2g_1)[G, G] = g_2g_1[G, G] = (g_2[G, G])(g_1[G, G])$. Thus $G/[G, G]$ is an abelian group. Note $f(g_1g_2) = f(g_1)f(g_2) = f(g_2)f(g_1) = f(g_2g_1)$ and $f([G, G]) = 0$. Since $[G, G] \subset \ker(f)$, there is a unique abelian group homomorphism $h : G/[G, G] \to H$ such that $ph = f$ for projection $p : G \to G/[G, G]$.

We will define the functor $L : \mathcal{C} \to \mathcal{C}'$ as $L(G) := G/[G, G]$ for $[G, G]$ the commutator subgroup. Note that a morphism of groups $f : G \to H$ gives a unique morphism $\overline{f} : G \to H/[H, H]$ by composing with the projection. Since $H/[H, H]$ is an abelian group, the above argument implies $\overline{f}$ factors uniquely through $G/[G, G]$ as $\overline{f} = pg$ for $p : G \to [G, G]$ the projection. Note that $g(a[G, G]) = f(a)[H, H]$ for $a \in G$. Define $L(f) := g$. Let $1_G : G \to G$ be the identity group homomorphism. Then $\overline{1_G} : G \to G/[G, G]$ factors uniquely as the identity on $G/[G, G]$. We have $L(1_G) = 1_{L(G)}$. Now let $f : G \to H$ and $g : H \to I$ be two group homomorphisms. Then $gf : G \to I$ gives $L(gf) = h$ for $h : G/[G, G] \to I/[I, I]$ an abelian group homomorphism defined as $h(a[G, G]) = (g(f(a))[I, I]$. Now $L(f) : G/[G, G] \to H/[H, H]$ gives $L(f)(a[G, G]) = f(a)[H, H]$ and $L(g) : H/[H, H] \to I/[I, I]$ gives $L(g)(f(a)[H, H]) = g(f(a))[I, I]$. Thus $L(gf) = L(g)L(f)$ and L is a covariant functor.

We want to show that $\mathrm{Hom}_{\mathcal{C}}(A, F(B))$ and $\mathrm{Hom}_{\mathcal{C}'}(L(A), B)$ are in bijective correspondence for $A \in \mathrm{Ob}(\mathcal{C})$ and $B \in \mathrm{Ob}(\mathcal{C}')$ and the bijection is functorial in A and B. As we have seen, some $f \in \mathrm{Hom}_{\mathcal{C}}(A, F(B))$ factors uniquely through $L(A) = A/[A, A]$ since B is an abelian group. Define the natural isomorphism Φ whereby $\Phi_{A,B}(f)$ is this unique morphism. Thus $\mathrm{Hom}_{\mathcal{C}}(A, F(B)) \simeq \mathrm{Hom}_{\mathcal{C}'}(L(A), B)$ via $\Phi_{A,B}$. Let $g : A' \to A$ be a morphism of groups. Then we want to show the diagram below commutes. Note that $g([A', A']) \subset [A, A] = \ker(A \to A/[A, A])$ so $L(g)$ factors uniquely through $A'/[A', A']$. We note that $L(g) : A'/[A', A'] \to A/[A, A]$ is this unique morphism. Then $\Phi_{A,B}(f) \circ L(g) : A'/[A', A'] \to B$ descends from $f \circ g : A' \to A \to B$. By construction, $\Phi_{A',B}(f \circ g)$ descends from $f \circ g$. The uniqueness of these morphisms implies $\Phi_{A,B}(f) \circ L(g) = \Phi_{A',B}(f \circ g)$ and we are functorial in A. A similar argument shows the bijection is functorial in B. We conclude that L is a left adjoint to F.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(A, F(B)) \arrow[r, "\Phi_{A,B}"] \arrow[d, "-\circ g"] & \mathrm{Hom}_{\mathcal{C}'}(L(A), B) \arrow[d, "-\circ L(g)"] \\
\mathrm{Hom}_{\mathcal{C}}(A', F(B)) \arrow[r, "\Phi_{A',B}"] & \mathrm{Hom}_{\mathcal{C}'}(L(A'), B)
\end{tikzcd}$$

We will show that F does not have a right adjoint. We will first prove that a left adjoint functor F preserves coproducts. Let G be the right adjoint. Let A_i be objects of $\mathcal{C}$ and B an object of $\mathcal{D}$. Then

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}\left(F\left(\coprod_i A_i\right), B\right) &\simeq \mathrm{Hom}_{\mathcal{D}}\left(\coprod_i A_i, B\right) \\
&\simeq \prod_i \mathrm{Hom}_{\mathcal{D}}(A_i, G(B)) \\
&\simeq \prod_i \mathrm{Hom}_{\mathcal{C}}(F(A_i), B) \\
&\simeq \mathrm{Hom}_{\mathcal{C}}\left(\coprod_i F(A_i), B\right).
\end{aligned}$$

By Yoneda Lemma, $F\left(\coprod_i A_i\right) \simeq \coprod_i F(A_i)$. The coproduct in the category of groups is the free product while the coproduct in the category of abelian groups is the direct sum. The free product $\mathbb{Z} * \mathbb{Z}$ is not isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$ so F does not have a right adjoint.

Fall 2017 Problem 10. Let $\mathcal{C}$ be a category with finite products, and let $\mathcal{C}^2$ be the category of pairs of objects of $\mathcal{C}$ together with morphisms $(A, A') \to (B, B')$ of pairs consisting of pairs $(A \to B, A' \to B')$ of morphisms in $\mathcal{C}$. Let $F : \mathcal{C}^2 \to \mathcal{C}$ be the direct product functor (that takes pairs of objects and morphisms to their products).

(a) Find a left adjoint to F.

Let $C, D \in \mathrm{Ob}(\mathcal{C})$ and $f \in \mathrm{Hom}_{\mathcal{C}}(C, D)$. Define $L : \mathcal{C} \to \mathcal{C}^2$ as $L(C) := (C, C)$ and $L(f) : L(C) \to L(D)$ as (f, f). Then $L(1_C) = (1_C, 1_C) = 1_{L(C)}$. Additionally, $L(gf) = (gf, gf) = (g, g) \circ (f, f) = L(g)L(f)$ for a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(D, E)$ and $E \in \mathrm{Ob}(\mathcal{C})$. Thus L is a functor.

By the universal property of the direct product, there is a unique morphism $h : C \to A \prod B$ for each pair of morphisms $(f, g) : (C, C) \to (A, B)$ such that $\pi_A \circ h = f$ and $\pi_B \circ h = g$. Define a natural transformation

$\Phi : \mathrm{Hom}_{\mathcal{C}^2}(L(-), -) \to \mathrm{Hom}_{\mathcal{C}^2}(-, F(-))$ so that $\Phi_{C,(A,B)} : \mathrm{Hom}_{\mathcal{C}}(L(C), (A,B)) \to \mathrm{Hom}_{\mathcal{C}^2}(C, F(A,B))$ gives $\Phi_{C,(A,B)}(f,g) := h$. Let $k \in \mathrm{Hom}_{\mathcal{C}}(C', C)$ for $C' \in \mathrm{Ob}(\mathcal{C})$. We want to show the following diagram commutes.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(L(C),(A,B)) \arrow[r, "\Phi_{C,(A,B)}"] \arrow[d, "(-\circ k, -\circ k)"] & \mathrm{Hom}_{\mathcal{C}^2}(C, F(A,B)) \arrow[d, "-\circ k"] \\
\mathrm{Hom}_{\mathcal{C}}(L(C'),(A,B)) \arrow[r, "\Phi_{C',(A,B)}"] & \mathrm{Hom}_{\mathcal{C}^2}(C', F(A,B))
\end{tikzcd}$$

Let $(f,g) \in \mathrm{Hom}_{\mathcal{C}}(L(C),(A,B)) = \mathrm{Hom}_{\mathcal{C}}((C,C),(A,B))$. We have $\Phi_{C,(A,B)}(f,g) \circ k$ is a morphism from C' to $A \prod B$ for which $\pi_A \circ (\Phi_{C,(A,B)}(f,g) \circ k) = f \circ k$ and $\pi_B \circ (\Phi_{C,(A,B)}(f,g) \circ k) = g \circ k$. Further, $h := \Phi_{C',(A,B)}(f \circ k, g \circ k)$ is the unique morphism $C' \to A \prod B$ that commutes with $f \circ k$ and $g \circ k$ under projection morphisms.

$$\begin{tikzcd}
C' \arrow[dr, dashed, "h"] \arrow[drr, bend left, "g\circ k"] \arrow[ddr, bend right, "f\circ k"'] & & \\
& A \prod B \arrow[r, "\pi_B"] \arrow[d, "\pi_A"] & B \\
& A &
\end{tikzcd}$$

Thus the universal property of the direct product implies $\Phi_{C,(A,B)}(f,g) \circ k = \Phi_{C',(A,B)}(f \circ k, g \circ k)$ and the desired diagram commutes. By a similar argument, we obtain naturality in (A,B). We conclude that L is a left adjoint to F.

(b) For $\mathcal{C}$ the category of abelian groups, determine whether or not F has a right adjoint.

Since abelian groups is an abelian category, finite products and coproducts are isomorphic. Define $R : \mathcal{C} \to \mathcal{C}^2$ as $R(C) := (C,C)$ and $R(f) := (f,f)$ for $f \in \mathrm{Hom}_{\mathcal{C}}(C,D)$. Then $R(1_C) = (1_C, 1_C) = 1_{R(C)}$. Additionally, $R(gf) = (gf, gf) = (g,g) \circ (f,f) = R(g)R(f)$ for a morphism $g \in \mathrm{Hom}_{\mathcal{C}}(D,E)$ and $E \in \mathrm{Ob}(\mathcal{C})$. Thus R is a functor.

By the universal property of the coproduct, there is a unique morphism $h : A \coprod B \to C$ for each pair $(f,g) : (A,B) \to (C,C)$ such that $h \circ i_A = f$ and $h \circ i_B = g$. Define the natural transformation $\Phi : \mathrm{Hom}_{\mathcal{C}^2}(-, R(-)) \to \mathrm{Hom}_{\mathcal{C}}(F(-), -)$ as $\Phi_{(A,B),C}(f,g) := h$. As in (a), the universal property of the coproduct implies naturality in (A,B) and C. We conclude that R is a right adjoint to F.

Fall 2016 Problem 8. Prove that if a functor $F : \mathcal{C} \to \mathrm{Sets}$ has a left adjoint functor, then F is representable.

Let $L : \mathrm{Sets} \to \mathcal{C}$ be the left adjoint to F. Then we know that $\Phi_{A,B} : \mathrm{Hom}_{\mathcal{C}}(L(A), B) \simeq \mathrm{Hom}_{\mathrm{Sets}}(A, F(B))$ for some natural isomorphism Φ and $A \in \mathrm{Ob}(\mathrm{Sets})$ and $B \in \mathrm{Ob}(\mathcal{C})$. Let $A := \{*\}$ be a set with one element. Then $\mathrm{Hom}_{\mathrm{Sets}}(A, F(B)) \simeq F(B)$ as sets via the morphism $h_B : \mathrm{Hom}_{\mathrm{Sets}}(A, F(B)) \to F(B)$ with $h_B(\alpha) := \alpha(*)$. Thus $\mathrm{Hom}_{\mathcal{C}}(L(A), B) \simeq \mathrm{Hom}_{\mathrm{Sets}}(A, F(B)) \simeq F(B)$ for all $B \in \mathrm{Ob}(\mathcal{C})$.

Define a natural transformation $\eta_B : \mathrm{Hom}_{\mathcal{C}}(L(A), B) \to F(B)$ by $\eta_B(f) := \Phi_{A,B}(f)(*)$. Since $\Phi_{A,B}$ is an isomorphism and $\mathrm{Hom}_{\mathrm{Sets}}(A, F(B)) \simeq F(B)$ by choosing the image of $* \in A$, we conclude that η_B is an isomorphism for each $B \in \mathrm{Ob}(\mathcal{C})$. Let $f \in \mathrm{Hom}_{\mathcal{C}}(L(A), B)$, and let $g : B \to C$ be a morphism in $\mathcal{C}$ for $C \in \mathrm{Ob}(\mathcal{C})$. We want to show the diagram below commutes. Since Φ is a natural transformation, the square on the left commutes. The square on the right commutes since $F(g)(h_B(\alpha)) = F(g)(\alpha(*))$ and $h_C(F(g) \circ \alpha) = (F(g) \circ \alpha)(*)$ for $\alpha \in \mathrm{Hom}_{\mathrm{Sets}}(A, F(B))$. Therefore, the diagram commutes. We conclude that F is represented by $L(A) \in \mathrm{Ob}(\mathcal{C})$.

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(L(A), B) \arrow[r, "\Phi_{A,B}"] \arrow[d, "g\circ -"] \arrow[rr, bend left, "\eta_B"] & \mathrm{Hom}_{\mathrm{Sets}}(A, F(B)) \arrow[r, "h_B"] \arrow[d, "F(g)\circ -"] & F(B) \arrow[d, "F(g)"] \\
\mathrm{Hom}_{\mathcal{C}}(L(A), C) \arrow[r, "\Phi_{A,C}"] \arrow[rr, bend right, "\eta_C"'] & \mathrm{Hom}_{\mathrm{Sets}}(A, F(C)) \arrow[r, "h_C"] & F(C)
\end{tikzcd}$$

Definition 8. An *initial object* of a category $\mathcal{C}$ is an object I such that, for every object X of $\mathcal{C}$, there exists one and only one morphism $I \to X$. A *terminal object* of a category $\mathcal{C}$ is an object T such that, for every object X of $\mathcal{C}$, there is one and only one morphims $X \to T$.

Spring 2016 Problem 2. Consider the functor F from commutative rings to abelian groups that takes a commutative ring R to the group R^* of invertible elements. Does F have a left adjoint? Does F have a right adjoint? Justify your answers.

We will show that F has a left adjoint. Define the functor $L : \text{Ab} \to \text{CRing}$ as $L(A) = \mathbb{Z}[A]$, the group ring over $\mathbb{Z}$. For an abelian group morphism $f : X \to Y$, we define $L(f) : \mathbb{Z}[X] \to \mathbb{Z}[Y]$ as $L(f)(x) = f(x)$ and extend $\mathbb{Z}$-linearly. Note that $L(f)$ is well-defined since $x \in X$ is a unit in $\mathbb{Z}[X]$ and it maps to a unit in $\mathbb{Z}[Y]$. Additionally, $L(f)$ is a unique commutative ring morphism that agrees with f on X since $\mathbb{Z}$ is initial in CRings. Let $1_X : X \to X$ be the identity morphism. Then $L(1_X)(\sum_{x\in X} a_x x) = \sum_{x\in X} a_x x$ and $L(1_X) = 1_{L(X)}$ for $a_x \in \mathbb{Z}$. Let $f : X \to Y$ and $g : Y \to Z$ be two abelian group morphisms. Then $L(gf)(\sum_{x\in X} a_x x) = \sum_{x\in X} a_x g(f(x)) = L(g)(\sum_{x\in X} a_x f(x)) = L(g)(L(f)(\sum_{x\in X} a_x x))$ for $a_x \in \mathbb{Z}$. Thus $L(gf) = L(g)L(f)$ and L is a functor.

We want to show that L is a left adjoint to F. Let $f : A \to F(B)$ be an abelian group morphism for $A \in \text{Ob(Ab)}$ and $B \in \text{Ob(CRing)}$. Define a natural transformation $\Phi_{A,B} : \text{Hom}_{\text{Ab}}(A, F(B)) \to \text{Hom}_{\text{CRing}}(L(A), B)$ by $\Phi_{A,B}(f)(x) = f(x)$ and extend $\mathbb{Z}$-linearly. By above, this is well-defined and the unique commutative ring morphism that agrees with f on X. Since units must map to units in a commutative ring morphism, every $h \in \text{Hom}_{\text{CRing}}(L(A), B)$ restricts to a morphism in $\text{Hom}_{\text{Ab}}(A, F(B))$. Thus $\Phi_{A,B}$ is a bijection. We want to show that the bijection is functorial in A and B. Let $g : A' \to A$ be a morphism of abelian groups. We want the diagram below to commute. Let $f \in \text{Hom}_{\text{Ab}}(A, F(B))$ as before. Then $\Phi_{A,B}(f) \circ L(g) : L(A') \to B$ extends the morphism $f \circ g : A' \to F(B)$. By definition, $\Phi_{A',B}(f \circ g)$ is also a morphism that extends $f \circ g$. The uniqueness in our choices of this morphism implies $\Phi_{A,B}(f) \circ L(g) = \Phi_{A',B}(f \circ g)$ and the diagram commutes. The argument for B is similar so the bijection is functorial in A and B. Therefore, L is a left adjoint to F.

$$\begin{array}{ccc}
\text{Hom}_{\text{Ab}}(A, F(B)) & \xrightarrow{\Phi_{A,B}} & \text{Hom}_{\text{CRing}}(L(A), B) \\
\big\downarrow{\scriptstyle -\circ g} & & \big\downarrow{\scriptstyle -\circ L(g)} \\
\text{Hom}_{\text{Ab}}(A', F(B)) & \xrightarrow{\Phi_{A',B}} & \text{Hom}_{\text{CRing}}(L(A'), B)
\end{array}$$

We will now show that left adjoints preserve initial objects. Let $L : \mathcal{C} \to \mathcal{D}$ and $R : \mathcal{D} \to \mathcal{C}$ be an adjoint pair. Let $A \in \text{Ob}(\mathcal{C})$ be an initial object. Then $\text{Hom}_{\mathcal{D}}(L(A), B) \simeq \text{Hom}_{\mathcal{C}}(A, R(B))$ for any $B \in \text{Ob}(\mathcal{D})$. But A initial in $\mathcal{C}$ implies $\text{Hom}_{\mathcal{C}}(A, R(B))$ has only one element. We conclude that $\text{Hom}_{\mathcal{D}}(L(A), B)$ has only one element and $L(A)$ is initial in $\mathcal{D}$. We want to show that F does not have a right adjoint. We note that $\mathbb{Z}$ is initial in CRings, but $F(\mathbb{Z}) \simeq \{\pm 1\} \simeq \mathbb{Z}/2\mathbb{Z}$ since ± 1 are the only units in $\mathbb{Z}$. The abelian group $\mathbb{Z}/2\mathbb{Z}$ is not initial since $\text{Hom}_{\text{Ab}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ has two elements, the trivial morphism and an isomorphism. Thus F cannot have a right adjoint.

Definition 9. Alternatively, F and G form an *adjunction pair* with F a *left adjoint* to G and G a *right adjoint* to F if there are natural transformations $\varepsilon : FG \to 1_{\mathcal{C}}$ and $\eta : 1_{\mathcal{D}} \to GF$ such that

$$F \xrightarrow{F\eta} FGF \xrightarrow{\varepsilon F} F$$

$$G \xrightarrow{\eta G} GFG \xrightarrow{G\varepsilon} G$$

are the identity transformations on F and G respectively. We call ε the *counit* and η the *unit.*

To derive the unit and counit from our earlier definition, note that an adjoint pair $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ defines a natural transformation of functors

$$\text{Hom}_{\mathcal{D}}(F(-), -) \to \text{Hom}_{\mathcal{C}}(-, G(-)).$$

For an object X in $\mathcal{C}$, we obtain a morphism

$$\text{Hom}_{\mathcal{D}}(F(X), F(X)) \to \text{Hom}_{\mathcal{C}}(X, G(F(X))).$$

that sends $\text{id}_{F(X)}$ to a morphism $X \to G(F(X))$. There is a similar setup for an object Y in $\mathcal{D}$. These adjunction maps are functorial in X and Y so we obtain the unit and counit described above.

Definition 10. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor of locally small categories. Define the set map

$$\varphi_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$$

to be $\varphi_{X,Y}(f) = F(f)$ for any two objects X and Y of $\mathcal{C}$. The functor F is *faithful* if $\varphi_{X,Y}$ is injective for each pair of objects. The functor F is *full* if $\varphi_{X,Y}$ is surjective for each pair of objects.

Fall 2018 Problem 7. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor with a right adjoint G. Show that F is fully faithful if and only if the unit of the adjunction $\eta : \mathrm{Id}_{\mathcal{C}} \to GF$ is an isomorphism.

Let $\varepsilon : FG \to 1_{\mathcal{D}}$ be the counit of the adjunction. ($\Rightarrow$) Assume F is fully faithful. We will show that $\eta_Y : Y \to GF(Y)$ is an isomorphism. Let $f, g : X \to Y$ be morphisms in $\mathcal{C}$ such that $\eta_Y \circ f = \eta_Y \circ g$. By the adjunction, $\mathrm{Hom}_{\mathcal{C}}(X, GF(Y)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ so $\eta_Y \circ f$ and $\eta_Y \circ g$ map to the same morphism $h : F(X) \to F(Y)$. Since F is fully faithful, $F_{X,Y} : \mathrm{Hom}_{\mathcal{C}}(X, Y) \to \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$. Thus $f = g$ and η_Y is left cancellative. Since F is full, we have $h : GF(X) \to X$ such that $F(h) = \varepsilon_{F(X)}$ for each $X \in \mathrm{Ob}(\mathcal{C})$. Then

$$\varepsilon_{F(X)} \circ F(\eta_X \circ h) = (\varepsilon_{F(X)} \circ F(\eta_X)) \circ F(h) = F(h) = \varepsilon_{F(X)} = \varepsilon_{F(X)} \circ F(1_X)$$

for all $X \in \mathrm{Ob}(\mathcal{C})$. Note that F is faithful so $\eta_X \circ h = 1_X$ and η_X is right cancellative. We conclude η is an isomorphism.

($\Leftarrow$) Assume η is an isomorphism. Let $f \in \mathrm{Hom}_{\mathcal{C}}(X, Y)$. Since η_Y is an isomorphism, $\eta_Y \circ -$ is a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, GF(Y))$. Via the adjunction, $\varepsilon_{F(Y)} \circ F(\eta_Y \circ f) = \varepsilon_{F(Y)} \circ F(\eta_Y) \circ F(f) = F(f)$. As a result, $\mathrm{Hom}_{\mathcal{C}}(X, Y) \simeq \mathrm{Hom}_{\mathcal{C}}(X, GF(Y)) \simeq \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y))$ via $F_{X,Y}$ and F is fully faithful.

$$\begin{tikzcd} X \arrow[d, "f"] \arrow[dr, "\eta_Y \circ f"] & \\ Y \arrow[r, "\eta_Y"] & GF(Y) \end{tikzcd} \qquad \begin{tikzcd} & F(X) \arrow[d, "F(\eta_Y \circ f)"] \arrow[dr, bend left, "F(f)"] & \\ F(Y) \arrow[r, "F(\eta_Y)"] & FGF(Y) \arrow[r, "\varepsilon_{F(Y)}"] & F(Y) \end{tikzcd}$$

Definition 11. A *monomorphism* is a left-cancellative morphism. In other words, f is a monomorphism if

$$(f \circ g_1 = f \circ g_2) \Rightarrow (g_1 = g_2)$$

for all suitable morphisms g_1 and g_2. An *epimorphism* is a right-cancellative morphism. In this case, g is an epimorphism if

$$(f_1 \circ g = f_2 \circ g) \Rightarrow (f_1 = f_2)$$

for all suitable morphisms f_1 and f_2.

Monomorphisms are the categorical analog of injective functions while epimorphisms are the categorical analog of surjective functions.

Fall 2015 Problem 1. Show that the inclusion $\mathbb{Z} \to \mathbb{Q}$ is an epimorphism in the category of rings with multiplicative identity.

We want to show that $f : \mathbb{Z} \to \mathbb{Q}$ is right cancellative. Let $g, h : \mathbb{Q} \to R$ be ring homomorphisms such that $gf = hf$ for R a ring with identity. For $a, b \in \mathbb{Z}$ we have

$$g\left(\frac{a}{b}\right) = g(a)g(b^{-1}) = g(a)g(b)^{-1} = h(a)h(b)^{-1} = h\left(\frac{a}{b}\right)$$

since $g(a) = g(f(a)) = h(f(a)) = h(a)$ for all $a \in \mathbb{Z}$. We conclude $g = h$ and f is an epimorphism

Spring 2018 Problem 5. Let $\mathcal{C}$ be a category. A morphism $f : A \to B$ in $\mathcal{C}$ is called an epimorphism if for any two morphisms $g, h : B \to X$ in $\mathcal{C}$, $g \circ f = h \circ f$ implies $g = h$. Let ALG be the category of $\mathbb{Z}$-algebras, and let MOD be the category of $\mathbb{Z}$-modules.

(a) Prove that in MOD, $f : M \to N$ is an epimorphism if and only if f is a surjection.

($\Rightarrow$) We will prove the contrapositive. Assume that $f : M \to N$ is not surjective. Then $\mathrm{im}(f) \subset N$ is a proper $\mathbb{Z}$-submodule. We define $\pi : N \to N/\mathrm{im}(f)$ the canonical projection and $g : N \to N/\mathrm{im}(f)$ the zero $\mathbb{Z}$-module homomorphism. Then gf and πf are zero maps so $gf = \pi f$. Let $n \in N$ such that $n \notin \mathrm{im}(f)$. Then $g \neq \pi$ since $g(n) = 0 + im(f)$ while $g(n) = n + \mathrm{im}(f) \neq 0 + \mathrm{im}(f)$. We conclude that f is not an epimorphism

($\Leftarrow$) Let $f : M \to N$ be surjective. Let $g, h : N \to P$ be $\mathbb{Z}$-module homomorphisms such that $gf = hf$. Let $n \in N$, then $n = f(m)$ for some $m \in M$. As a result, $g(n) = g(f(m)) = h(f(m)) = h(n)$ so $g = h$. We conclude that f is right-cancellative and f is an epimorphism.

(b) In ALG, does the equivalence of epimorphism and surjection hold? If yes, prove the equivalence, and if no, give a counterexample (as simple as possible).

Let $i : \mathbb{Z} \to \mathbb{Q}$ be the canonical inclusion morphism of $\mathbb{Z}$-algebras. By Fall 2015 Problem 1, this morphism is a non-surjective epimorphism.

In more recent years, there have been problems about abelian categories. The prototypical example of an abelian category is R-Mod for a ring R. If the finite product and finite coproduct constructions are isomorphic in a category, we refer to the operation as a direct sum. As an example, see the direct sum in Ab.

Definition 12. An object that is both initial and terminal is a *zero object.*

Definition 13. An *additive category* is a category admitting a zero object, any two pairs of objects admits a direct sum, and every Hom set has an abelian group structure.

Definition 14. The *kernel* of a morphism $f : X \to Y$ is an object $\ker(f)$ together with a morphism $i : \ker(f) \to X$ that satisfies the following universal property. If there is a morphism $g_X : Z \to X$ such that $f \circ g = 0_{Z,Y}$, then there is a unique morphism $u : Z \to \ker(f)$ such that $g = i \circ u$.

$$\begin{tikzcd}
 & X \arrow[dr, "f"] & \\
 & \ker(f) \arrow[u, "i"] \arrow[r, "0"] & Y \\
Z \arrow[uur, bend left, "g"] \arrow[ur, dashed, "u"] \arrow[urr, bend right, "0"'] & &
\end{tikzcd}$$

The dual notion (where we flip the arrows) is a *cokernel* of $f : X \to Y$ denoted $\mathrm{coker}(f)$.

In a category like R-Mod, the categorical kernel of a morphism $f : X \to Y$ is an equivalent notion to that of our standard element-wise kernel. Further, the cokernel can be thought of as $Y/\mathrm{im}(f)$.

Definition 15. An *abelian category* is an additive category in which each morphism has a kernel and cokernel and, for each $f : X \to Y$, the canonical morphism $\mathrm{coker}(\ker(f)) \to \ker(\mathrm{coker}(f))$ is an isomorphism.

The purpose of abelian categories is that it is the most general setting in which we can discuss exact sequences. Homological algebra is the study of abelian categories.

Definition 16. An object P is *projective* if for any epimorphism $e : E \to X$ and morphism $f : P \to X$, there is a morphism $g : P \to E$ such that $e \circ g = f$.

$$\begin{tikzcd}
 & P \arrow[d, "f"] \arrow[dl, dashed, "g"'] \\
E \arrow[r, two heads, "e"] & X
\end{tikzcd}$$

Definition 17. An object P is *injective* if for any monomorphism $m : X \to Y$ and morphism $g : X \to Q$, there is a morphism $h : Y \to Q$ such that $h \circ m = g$.

$$\begin{tikzcd} X \arrow[r, hook, "m"] \arrow[d, "g"'] & Y \arrow[dl, dashed, "h"] \\ Q & \end{tikzcd}$$

Example 4. In the category of abelian groups, the projective objects are free abelian groups. The injective objects in the category of abelian groups are necessarily divisible. Assuming the axiom of choice, every divisible group is injective.

Spring 2019 Problem 10. Let $\mathcal{C}$ be an abelian category. Prove TFAE:

(1) Every object of $\mathcal{C}$ is projective.

(2) Every object of $\mathcal{C}$ is injective.

(1) $\Rightarrow$ (2): Assume that every object is projective. Let $m : X \to Y$ be a monomorphism for which there is a morphism $g : X \to Q$. We can build the short exact sequence

$$0 \longrightarrow X \xrightarrow{f} Y \xrightarrow{q} C \longrightarrow 0$$

where $C = \mathrm{coker}(m)$. We have the diagram

$$\begin{tikzcd} & C \arrow[dl, dashed] \arrow[d, "1_C"] \\ Y \arrow[r, two heads, "q"] & C. \end{tikzcd}$$

The morphism $C \to Y$ guaranteed by C projective is a splitting. In an abelian category left and right split are equivalent so there is a morphism $s : Y \to X$ such that $s \circ m = 1_X$. Define $h = g \circ s$ and

$$h \circ m = (g \circ s) \circ m = g \circ (s \circ m) = g.$$

Thus Q is injective.

(1) $\Leftarrow$ (2): Make a similar argument since a short exact sequence will split with an injective first entry.